General Approach to Normal Frequencies

The previous example relied heavily on symmetry. We may not always be able to do that.

Hence we need a more general approach.

The key point is that for the normal modes both oscillators are at the same frequency.

i.e. $x_A = C \cos \omega t$ — same ω

$x_B = C' \cos \omega t$

In this situation if we have C, C' & ω unknown plus the equations of motion

$$\frac{d^2 x_A}{dt^2} + (\omega_0^2 + \omega_c^2) x_A - \omega_c^2 x_B = 0$$

$$\frac{d^2 x_B}{dt^2} + (\omega_0^2 + \omega_c^2) x_B - \omega_c^2 x_A = 0$$

you might be tempted to think there isn't enough information to make progress - ie 3 unknowns & 2 equations, but let's see what happens.

Substitute for x_A & x_B into the equations of motion and then we get

$$-\omega^2 C \cancel{\cos \omega t} + (\omega_0^2 + \omega_c^2) C \cancel{\cos \omega t} - \omega_c^2 C' \cos \omega t = 0$$

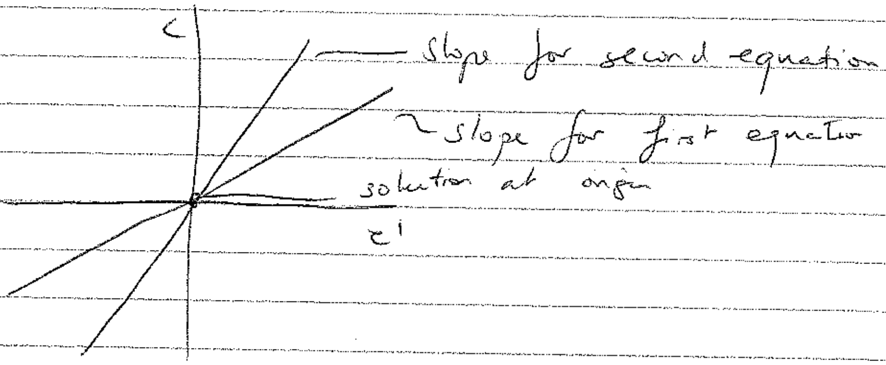
$$-\omega^2 C' \cancel{\cos \omega t} + (\omega_0^2 + \omega_c^2) C' \cancel{\cos \omega t} - \omega_c^2 C \cos \omega t = 0$$

Giving

$$(-\omega^2 + \omega_0^2 + \omega_c^2) C \qquad - \omega_c^2 C' = 0$$

$$-\omega_c^2 C \qquad + (-\omega^2 + \omega_0^2 + \omega_c^2) C' = 0$$

If we imagine C & C' as x & y in this case, then both these relationships pass through the origin, ie they're of the form $\alpha x = \beta y$, where α & β are given by the terms ahead of C & C'. You also know that if the coefficients are not multiples of one another then the only solution is the origin ie



So the interesting situation is when both sets of coefficients are equal & then the lines overlap:

ie the second equation is some multiple of the first!

If that's true C can take on any value but as soon as you specify it C' is determined.

The easiest way to proceed is to consider that the two slopes must be equal:

$$\frac{C}{C'} = \frac{\omega_c^2}{-\omega^2 + \omega_0^2 + \omega_c^2}$$ from the first equation

$$\frac{C}{C'} = \frac{-\omega^2 + \omega_0^2 + \omega_c^2}{\omega_c^2}$$ from the second equation

Equating the two:

$$\frac{(-w^2 + w_o^2 + w_c^2)}{w_c^2} = \frac{w_c^2}{(-w^2 + w_o^2 + w_c^2)}$$

hence we get

$$(w_c^2)^2 = (-w^2 + w_o^2 + w_c^2)^2$$

Taking square roots:

$$\pm w_c^2 = -w^2 + w_o^2 + w_c^2$$

Rearranging for w^2:

$$w^2 = w_o^2 + w_c^2 \pm w_c^2$$

Hence we get:

$$w^2 = w_o^2 \quad \text{or} \quad w_o^2 + 2w_c^2$$

frequency for "uncoupled" case

frequency for "coupled" case

Which is a quick & powerful way of determining the frequencies.

Q: What do we get for the amplitudes for these two solutions

A: Let's plug the values in!

Let $w^2 = w_o^2$

$$\Rightarrow \quad -w_c^2 C + w_c^2 C' = 0$$

from 2nd equation of motion

Thus $-C + C' = 0 \Rightarrow C = C'$

For $\omega^2 = \omega_0^2 + 2\omega_c^2$ we get

$$-\omega_c^2 C + -\omega_c^2 C' = 0$$

$$\Rightarrow -C - C' = 0 \Rightarrow C = -C'$$

Thus we've found both our solutions!

For $\omega = \omega_0$:

$$x_A = C\cos\omega_0 t \qquad x_B = C\cos\omega_0 t$$

Swinging together (same amplitude)

For $\omega^2 = \omega_0^2 + 2\omega_c^2 = \omega'^2$

$$x_A = D\cos\omega' t \qquad x_B = -D\cos\omega' t$$

D & -D means they oscillate in opposite directions.

Note we're using different amplitudes in both cases because the normal modes can have different amplitudes relative to one another

Since both cases represent solutions to the equations (and importantly the equations are linear in x_A & x_B) we can add:

$$x_A = C\cos\omega_0 t + D\cos\omega' t$$

$$x_B = C\cos\omega_0 t - D\cos\omega' t$$

Which is exactly what we had before!

This is a better approach in that we didn't need to make any assumptions about symmetry etc.

But we aren't quite finished - we're still missing the phase angles.

Must have same phases in each normal mode solution - otherwise it won't work. For example consider the situation where the spring is unstretched - if x_B has a different phase to x_A then the spring length has to change $\Rightarrow$ this is not allowed!

Let's call the lower frequency mode the "lower mode" & the higher frequency mode the "upper mode"

For obvious reasons (ie there is no additional spring force) the pendulum movement of the "lower mode" has a lower frequency

The oscillating nature of the "upper mode" means that there are additional forces involved & hence a higher frequency (ie from the spring).

Thus we can have

$$x_A = C\cos(\omega_0 t + \alpha)$$
$$x_B = C\cos(\omega_0 t + \alpha)$$

$$x_A = D\cos(\omega' t + \beta)$$
$$x_B = -D\cos(\omega' t + \beta)$$

and general solution

$$x_A = C\cos(\omega_0 t + \alpha) + D\cos(\omega' t + \beta)$$
$$x_B = C\cos(\omega_0 t + \alpha) - D\cos(\omega' t + \beta)$$

Q: How many constants can we set in these solutions?

A: 4

Corresponds to the initial position & velocity of A & B