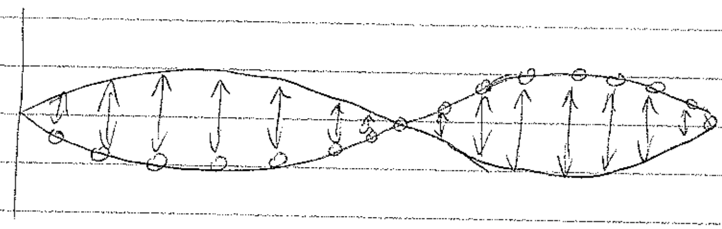
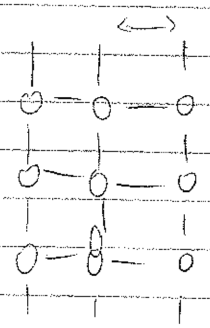
Coupled Oscillators

Why study? Because they're everywhere!

Biology – "pacemaker cells" in heart...
Chemistry – molecular bonds are springs...
Physics – vibrating strings, lattice solids...



Think of a vibrating string as many coupled oscillators



Oscillations of crystal lattice

Many examples are actually very subtle & complex. However, we can learn much by starting with a simple system & examining its properties

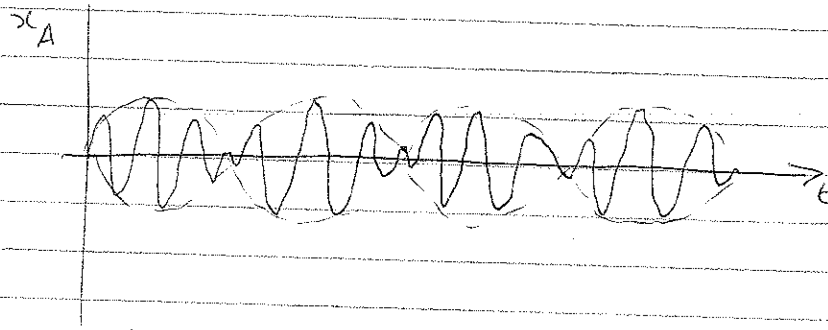
<u>2 coupled oscillators</u> – Pendulums coupled via a spring

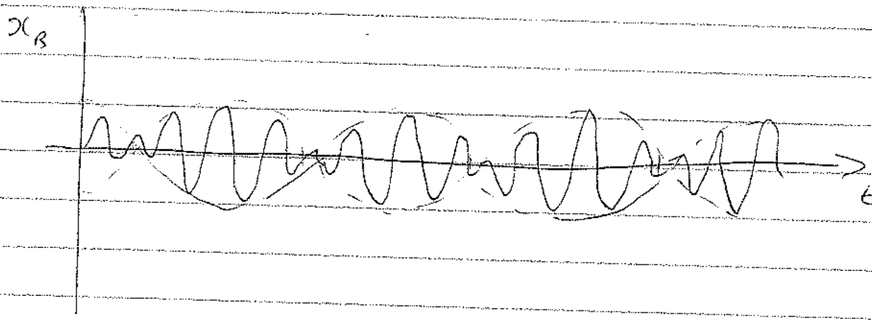
Demo: ① Set one oscillating while holding other still, let go.
Once second reaches max oscillation stop.

Q: What happens next? Vote!

② Restart to resolve
– Result: energy moves back & forth

What does this look like in terms of amplitude?

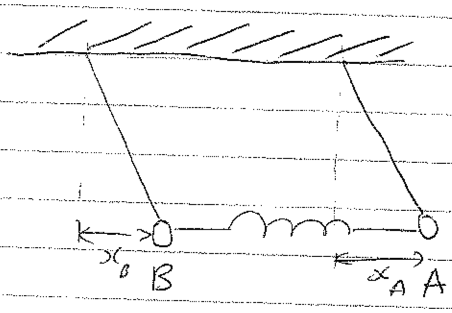




Should be clear that both sides are exhibiting beats - we'll show why that is as part of this lecture.

Q: How can we go about building up intuition?
A: Consider simple cases of oscillation first!

Easiest is if both pendulums oscillate together so spring is unstretched:



In this situation do they impact one another?

No!

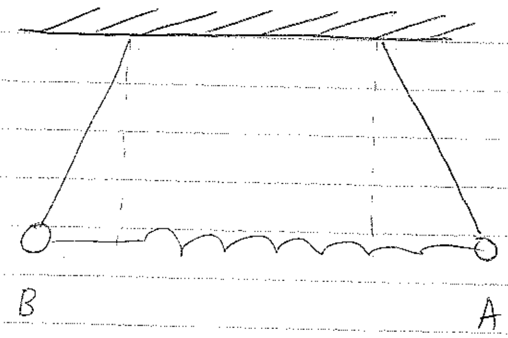
$$X_A = C \cos \omega_0 t$$

$$X_B = C \cos \omega_0 t$$

same amplitudes
same angular frequency

Nothing is transmitted through the spring.

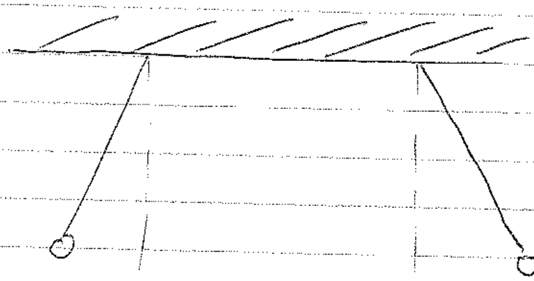
2nd easiest case:



Motion of A is the exact opposite of B

Spring repeatedly contract & expands.

To work out the oscillation behaviour here, first consider no spring case:



Forces are just from gravity

$$F = -kx = -m\omega_0^2 x$$

since $\omega_0 = \sqrt{\frac{g}{l}}$ for pendulum

So each oscillator has equation

$$m\ddot{x} + m\omega_0^2 x = 0 \qquad \text{whether } x_A \text{ or } x_B$$

If we put the spring in we must include an additional force

Extension $= 2x_A$ (by symmetry)

So extra force is $-2kx_A$, hence for x_A equation of motion

$$m\ddot{x}_A + m\omega_0^2 x_A = -2kx_A$$

$$\Rightarrow \quad m\ddot{x}_A + m\omega_0^2 x_A + 2kx_A = 0$$

$$\therefore \quad \ddot{x}_A + \left(\omega_0^2 + \frac{2k}{m}\right)x_A = 0$$

Let $\omega_c = \sqrt{\frac{k}{m}}$ Then

$$\ddot{x}_A + (\omega_0^2 + 2\omega_c^2)\, x_A = 0$$

$$\Rightarrow \quad \ddot{x}_A + \omega'^2 x_A = 0 \quad \text{if} \quad \omega'^2 = \omega_0^2 + 2\omega_c^2$$

So surprise! We've got SHM with a frequency of ω'^2, hence solution will be

$$x_A = D\cos(\omega' t)$$

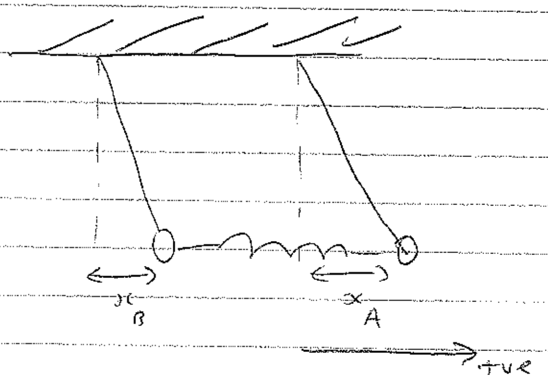
and by symmetry

$$x_B = -D\cos(\omega' t)$$

__But__ There is still no transfer of energy between the two oscillators. They carry on oscillating at ω' at the set amplitude.

These two modes are called the __NORMAL MODES__

Now examine the most general case:



Q: What is the spring extension given by?

$x_A - x_B$

Force on A

$$-\Big(\underbrace{m\omega_0^2 x_A}_{\text{gravity}} + \underbrace{k(x_A - x_B)}_{\text{spring}}\Big)$$

Force on B

$$-m\omega_0^2 x_B + k(x_A - x_B)$$

Thus

$$m\ddot{x}_A + m\omega_0^2 x_A + k(x_A - x_B) = 0 \qquad m\ddot{x}_B + m\omega_0^2 x_B - k(x_A - x_B) = 0$$

$$\ddot{x}_A + (\omega_0^2 + \omega_c^2)x_A - \omega_c^2 x_B = 0 \qquad \ddot{x}_B + (\omega_0^2 + \omega_c^2)x_B - \omega_c^2 x_A = 0$$

So now have two equations, but each has two unknowns $\Rightarrow$ need to solve simultaneously.

First though, consider the following approach.

ADD both equations together:

$$\ddot{x}_A + \ddot{x}_B + (\omega_0^2 + \cancel{\omega_c^2})x_A - \cancel{\omega_c^2 x_B} + (\omega_0^2 + \cancel{\omega_c^2})x_B - \cancel{\omega_c^2 x_A} = 0$$

$$\Rightarrow \quad \frac{d^2}{dt^2}(x_A + x_B) + \omega_0^2 (x_A + x_B) = 0$$

If we substitute $q_1 = x_A + x_B$

$$\Rightarrow \quad \ddot{q}_1 + \omega_0^2 q_1 = 0 \quad \text{— ie SHM for } q_1$$

If now we subtract $\ddot{x}_B$ equation from $\ddot{x}_A$ eqn:

$$\ddot{x}_A - \ddot{x}_B + (\omega_0^2 + \omega_c^2)x_A - \omega_c^2 x_B - (\omega_0^2 + \omega_c^2)x_B + \omega_c^2 x_A = 0$$

$$\Rightarrow \quad \frac{d^2}{dt^2}(x_A - x_B) + (\omega_0^2 + 2\omega_c^2)(x_A - x_B) = 0$$

let $q_2 = x_A - x_B$ Then

$$\ddot{q}_2 + \omega'^2 q_2 = 0 \quad \text{— ie SHM for } q_2$$

where $\omega'^2 = \omega_0^2 + 2\omega_c^2$

Thus we have two possible solutions for q_1 & q_2

$$\left.\begin{aligned} q_1 &= C \cos \omega_0 t \\ q_2 &= D \cos \omega' t \end{aligned}\right\} \text{ note we've set phases to zero here for simplicity}$$

Should be clear that these solutions relate to the normal modes (check the angular frequencies from earlier).

Q: How can we get back to x_1 & x_2 from these variables?

A: Easy to see that $x_A = \frac{1}{2}(q_1 + q_2)$

$$x_B = \frac{1}{2}(q_1 - q_2)$$

So if we substitute:

$$\left.\begin{aligned} x_A &= \frac{1}{2}\{C\cos\omega_0 t + D\cos\omega' t\} \\ x_B &= \frac{1}{2}\{C\cos\omega_0 t - D\cos\omega' t\} \end{aligned}\right\} \quad (\mathrm{I})$$

This is clearly a more general solution than what we considered before - it's actually a superposition of the solutions.

Let's see if we can recover the same behavior as in the demo.

At $t=0$ what will the initial conditions be?

$$\left.\begin{aligned} x_A &= A_0 \\ \dot{x}_A &= 0 \end{aligned}\right\}$$ start with spring extended on one side

$$\left.\begin{aligned} x_B &= 0 \\ \dot{x}_B &= 0 \end{aligned}\right\}$$ second mass starts at rest at eq/b position

Using the equations for x_A & x_B in (I) we get

$$x_A = A_0 = \frac{1}{2}\{C.1 + D.1\} = \frac{1}{2}(C+D)$$

$$x_B = 0 = \frac{1}{2}\{C.1 - D.1\} = \frac{1}{2}(C-D)$$

Add to show $A_0 = C$ and then

$$\Rightarrow A_0 = D$$

$$\therefore A_0 = C = D$$

Velocity values both work fine as $\sin\omega_0 t = 0 = \sin\omega' t$

Thus we get

$$x_A = \frac{A_0}{2}\left[\cos(\omega_0 t) + \cos(\omega' t)\right]$$

$$x_B = \frac{A_0}{2}\left[\cos(\omega_0 t) - \cos(\omega' t)\right]$$

Using $\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$

$\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$

we get

$$x_A = A_0 \cos\left[\frac{(\omega_0 - \omega')t}{2}\right]\cos\left[\frac{(\omega_0 + \omega')t}{2}\right]$$

$$x_B = -A_0 \sin\left[\frac{(\omega_0 - \omega')t}{2}\right]\sin\left[\frac{(\omega_0 + \omega')t}{2}\right]$$

So two beats-type solution which is exactly what we had earlier.

Two things to note:

(1) The lower frequency "envelope" functions are different, one sin, one cos.
But that exactly matches what we observed, ie one is max when other is minimum & vice versa

(2) The net frequency parts also have a phase shift.