Power in forced damped systems

Studying this concept is not just academic. Power usage in these systems is very relevant in the electrical analogue of the damped oscillator - the so called LCR circuit.

The analysis is, however, easier if we focus on the damped harmonic oscillator first.

First, lets define power in a form that is useful for us. Since power is rate of change of work, W, then

$$P = \frac{dW}{dt}$$

For the work, we have

$$W = \int F \, dx$$

where the integral is along the path of movement.

But if $v = \frac{dx}{dt}$ then we can rewrite

$$W = \int F v \, dt = \int F \dot{x} \, dt$$

Then for the power we find

$$P = \frac{d}{dt} \int F \dot{x} \, dt = F \dot{x}$$

ie instantaneous power input is given by the force times system velocity. We can now look at questions like "What rate must energy be fed into a driven oscillator to maintain its fixed amplitude?"*

As is usual, let's examine the undamped case first

Recall:

$$x = \frac{F_0/m}{\omega_0^2 - \omega^2} \cos \omega t$$

$$= C \cos \omega t$$ & ω is the frequency of the drive

By simply taking the derivative

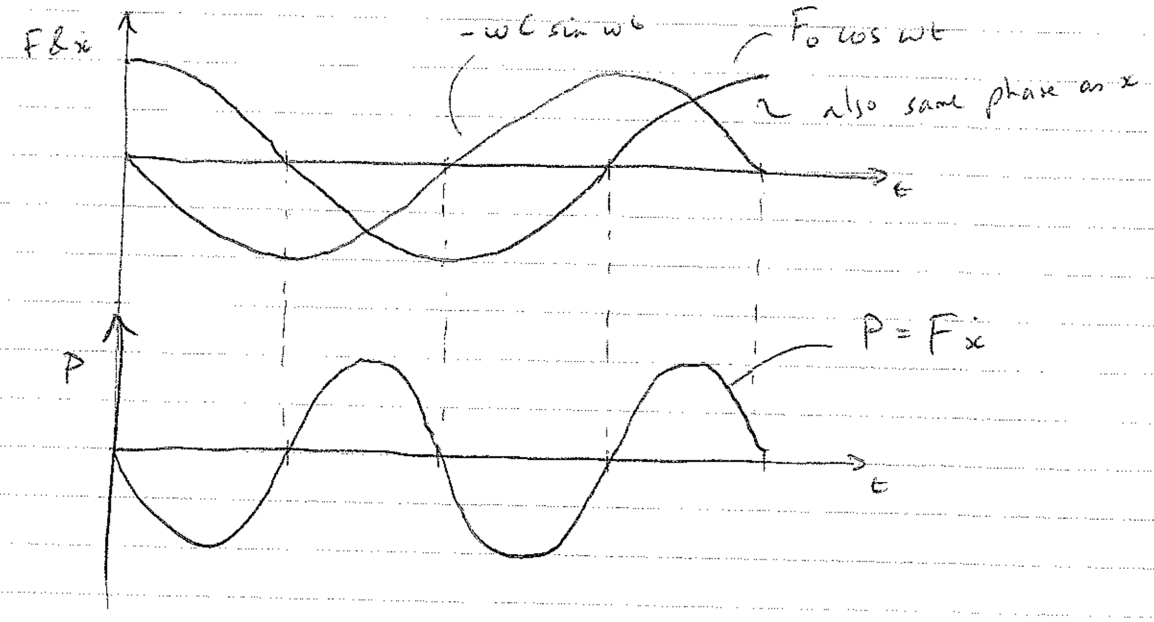
$$\dot{x} = -\omega C \sin \omega t$$

Hence we can write the power being used at a given time as

$$P = F\dot{x} = (F_0 \cos \omega t)(-\omega C \sin \omega t)$$

$$= -\frac{1}{2} \omega F_0 C \sin(2\omega t)$$ since $\sin 2A = 2 \cos A \sin A$

Plotting out terms:



Q: So on the basis of this graph what is the average power that needs to be put in of over 1 cycle? $(=T)$

A: Net power is zero!

Some times it is positive & sometimes it is negative $\Rightarrow$ some times put power in & sometimes get power out!

Intuitively this result may seem somewhat strange, as if one thinks about the sum of the kinetic & potential energies they sum to a constant number.

But the key point is to consider that as the system is driven it will impact on the driver itself via $F\dot{x}$. That's where the input or dissipation occurs, and ultimately it must average out to prevent the amplitude from growing.

The next step is to consider the damped case

Recall for the steady state we used a solution:

$$x = A\cos(\omega t - \delta)$$

$$\Rightarrow \quad \dot{x} = -\omega A \sin(\omega t - \delta)$$

where $$A = \frac{F_0}{k} \frac{\omega_0/\omega}{\left[\left(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}\right)^2 + \frac{1}{Q^2}\right]^{1/2}}$$

and $$\tan\delta = \frac{1/Q}{\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}}$$

Same approach as last time

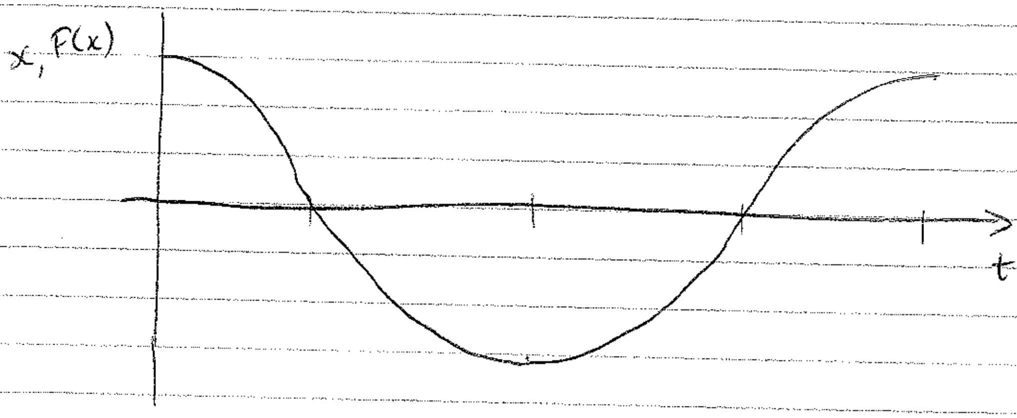
$$P = F\dot{x}$$

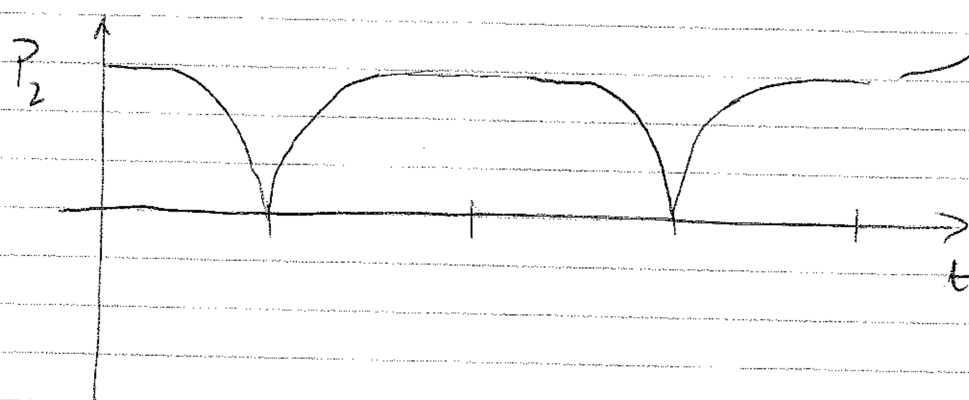
$$= F_0 \cos\omega t \cdot (-\omega A) \sin(\omega t - \delta)$$

$$= -F_0 \omega A \cos\omega t \left[\sin\omega t \cos\delta - \cos\omega t \sin\delta\right]$$

$$= -F_0 \omega A \cos\omega t \sin\omega t \cos\delta + F_0 \omega A \cos^2\omega t \sin\delta$$

$$= \underbrace{-\frac{F_0 \omega A}{2} \cos\delta \sin(2\omega t)}_{\text{Similar to what we saw before averages to zero over a cycle}} + \underbrace{F_0 \omega A \sin\delta \cos^2\omega t}_{\text{does not average to zero!} = P_2}$$





always +ve → always dissipation

Over one cycle what is the average power?

Recall we define the average of a function by

$$\langle A \rangle = \frac{\int_a^b A(x)\,dx}{b-a}$$

So we need to calculate

$$\frac{\int_0^{\frac{2\pi}{\omega}} \cos^2 \omega t \, dt}{\frac{2\pi}{\omega}}$$ since $F_0 \omega A \sin\delta$ is constant at given ω

Consider $\int_0^{\frac{2\pi}{\omega}} \cos^2 \omega t \, dt = \int_0^{2\pi} \cos^2 t' \frac{dt'}{\omega}$ where $t' = \omega t$

$$= \frac{1}{\omega}\int_0^{2\pi} \cos^2 t' \, dt'$$

$$= \frac{1}{\omega}\int_0^{2\pi} \frac{1}{2}(\cos 2t' + 1)\, dt'$$ since $\cos^2 x = \frac{1}{2}(\cos 2x + 1)$

$$= \frac{1}{\omega}\int_0^{2\pi} \frac{1}{2}\, dt' + \cancel{\frac{1}{2\omega}\int_0^{2\pi} \cos 2t' \, dt'}$$

$$= \frac{1}{\omega}\,\frac{2\pi}{2} = \frac{\pi}{\omega}$$

$$\therefore \langle \cos^2 \omega t \rangle = \frac{\pi/\omega}{2\pi/\omega} = \frac{1}{2}.$$

Thus average power over one cycle is

$$\bar{P} = \frac{1}{2} F_0 \omega A \sin\delta \qquad \text{sub for } A \text{ to give}$$

$$= \frac{1}{2} F_0 \cancel{\omega} \sin\delta \; \frac{F_0}{k} \; \frac{\omega_0 / \cancel{\omega}}{\left[\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)^2 + \frac{1}{Q^2}\right]^{1/2}}$$

But next consider how to express $\sin\delta$:

Since $\tan\delta = \dfrac{1/Q}{\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}} = \dfrac{\text{opp}}{\text{adj}}$

$$\sin\delta = \frac{\text{opp}}{\text{hyp}} = \frac{1/Q}{\left[\left(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}\right)^2 + \frac{1}{Q^2}\right]^{1/2}}$$

Substituting gives:

$$\bar{P} = \frac{1}{2} \frac{F_0^2 \omega_0}{kQ} \; \frac{1}{\left[\left(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}\right)^2 + \frac{1}{Q^2}\right]}$$

Q: When is $\bar{P}$ maximized? (In terms of ω?)

A: When $\omega_0 = \omega$ so denominator $\Rightarrow 1/Q^2$

If $\omega = \omega_0$ then can show

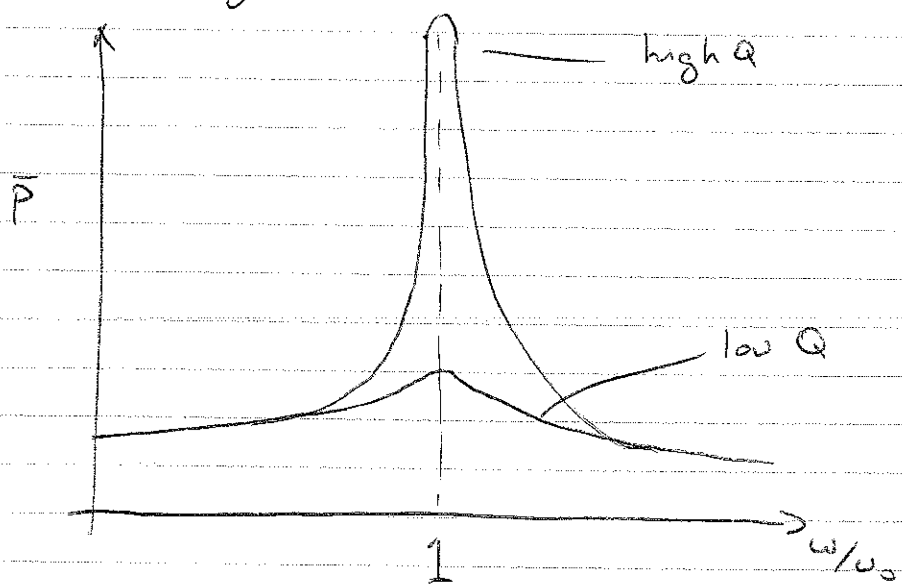
$$\bar{P}_{max} = \frac{1}{2} \frac{F_0^2 \omega_0}{kQ} \cdot Q^2 = \frac{F_0^2 \omega_0 Q}{2k} = \frac{Q F_0^2}{2m\omega_0} = \frac{F_0^2}{2b}$$

So have shown that

(1) Max power occurs at resonance

(2) It is dependent (linearly) on Q.

In general, behavior as a function of frequency looks as follows:



Read p. 101–112 for examples of resonances.
The LCR electrical circuit is especially important as an example of resonant systems.