Initial transients

When we considered free vibrations the phase & amplitudes were set by our initial conditions.

For the forced vibration case we found the phase & amplitude were completely set.

Q: What's different?

Not a property of the differential equation as they are both second order!

The answer is the trial solution for the driven system is not complete! There can be pieces of the solution that "vibrate" at the natural frequency & eventually damp away (assuming damping is in the problem).

You can thus think of the forced vibration solutions that we found as having been oscillating "for ever" i.e. any initial transients have damped away, so they weren't included.

Of course this picture is very incomplete.

- No physical system has been oscillating forever
- There is always a beginning state - we have not yet worked out how we can go from this situation to the "steady state".

Consider the situation with zero damping first.

For this situation $\ddot{x} + \omega_0^2 x = \frac{F_0}{m} \cos \omega t$

showed that we can solve with

$$x = A \cos \omega t = \frac{F_0/m}{\omega_0^2 - \omega^2} \cos \omega t$$

What if we want to have $\dot{x}=0$ at $t=0$ for this solution?

Since $$x = \frac{F_0/m}{\omega_0^2 - \omega^2} \cos \omega t$$

$$\Rightarrow \quad x = \frac{F_0/m}{\omega_0^2 - \omega^2} \times 1 \neq 0$$

So this solution is actually incompatible with setting $x=0$ at $t=0$.

You can't add an arbitrary phase either because the RHS of the equation depends upon $\cos \omega t$.

How can we handle this?

For the unforced oscillator we looked for solutions to

$$\ddot{x}_2 + \omega_0^2 x_2 = 0$$

If we were to add this to the forced situation

$$\ddot{x}_1 + \omega_0^2 x_1 = \frac{F_0}{m} \cos \omega t$$

what do we get?

$$\ddot{x}_2 + \ddot{x}_1 + \omega_0^2 x_2 + \omega_0^2 x_1 = \frac{F_0}{m} \cos \omega t$$

$$\Rightarrow \quad \frac{d^2}{dt^2}(x_2 + x_1) + \omega_0^2 (x_2 + x_1) = \frac{F_0}{m} \cos \omega t$$

If we let $x_2 + x_1 = x_3$ then

$$\ddot{x}_3 + \omega_0^2 x_3 = \frac{F_0}{m} \cos \omega t$$

Which is exactly the same equation as for x_1, but now it is in x_3.

$\Rightarrow$ x_3 is also a solution to the forced oscillator equation.

What does this mean?

It shows that the forced oscillator allow solutions that include components in both the oscillating (driving) frequency & the natural frequency of the system!!

In fact this is necessary to allow for all different possible starting conditions!

What happens physically?

Small initial impulse occurs first

$\Rightarrow$ puts system into oscillations at the natural frequency

$\Rightarrow$ but the continually applied force will support, solutions at angular frequency ω.

In many situations the natural frequency oscillations will die away, but early on the motion should show components of both parts.

[c.f. driven pendulum demo]

We've thus shown that the most general solution of the undamped forced oscillator is

$$x_3 = \underbrace{B\cos(\omega_0 t+\beta)}_{\text{free solution}} + \underbrace{C\cos\omega t}_{\text{steady state}} \qquad \text{where } C = \frac{F_0/m}{\omega_0^2-\omega^2}$$

The free solution amplitude & phase need to be specified.

Lets consider the situation where $\left.\begin{matrix} x=0 \\ \dot{x}=0 \end{matrix}\right\}$ at $t=0$.

so the system starts from rest.

$$x_3 = 0 \Rightarrow \quad 0 = B\cos(0+\beta) + C\cos 0$$

$$\Rightarrow \quad B\cos\beta + C = 0$$

$\dot{x}_3 = 0$ yields

$$\dot{x}_3 = -B\omega_0 \sin(\omega_0 t+\beta) - C\omega\sin(\omega t) = 0$$

but $t=0$ so

$$0 = -B\omega_0 \sin\beta - 0$$

$$\Rightarrow \quad \sin\beta = 0 \quad \text{so} \quad \beta = 0, \pi$$

Both $0, \pi$ solutions give the same result!

Take $\beta = 0$ for now then:

$$B\cos 0 + C = 0 \quad \text{gives} \quad B + C = 0$$
$$\Rightarrow C = -B$$

So we get a negative amplitude in this case.

Taking $\beta = \pi$ gives $\quad B\cos\pi + C = 0$
$$\Rightarrow -B + C = 0 \Rightarrow C = B$$

So an offset produces a +ve amplitude instead.

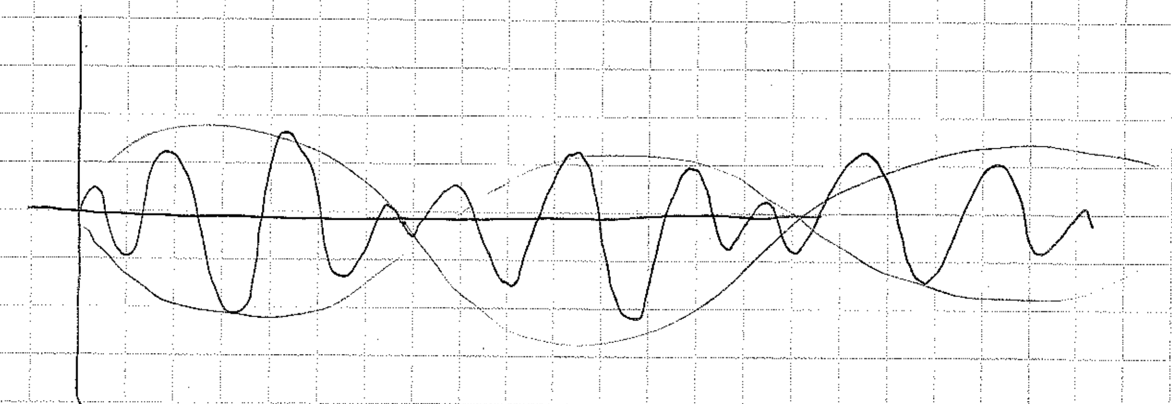
Hence we can write the solution as

$$x_3 = -C\cos\omega_0 t + C\cos\omega t$$
$$= C(\cos\omega t - \cos\omega_0 t)$$

Does this remind you of anything?
What if ω & ω_0 are close together?
System should show beats!
Can show using $\cos A - \cos B = -2\sin\frac{1}{2}(A+B)\sin\frac{1}{2}(A-B)$
that

$$x_3 = -2C\sin\left(\frac{\omega+\omega_0}{2}\right)\sin\left(\frac{\omega-\omega_0}{2}\right).$$

So we'll see something like



Which is exactly what we observed when starting the driven pendulum demo from rest!

Adding damping:

NO surprise that we need to consider the sum of the solution without driving & the one with.

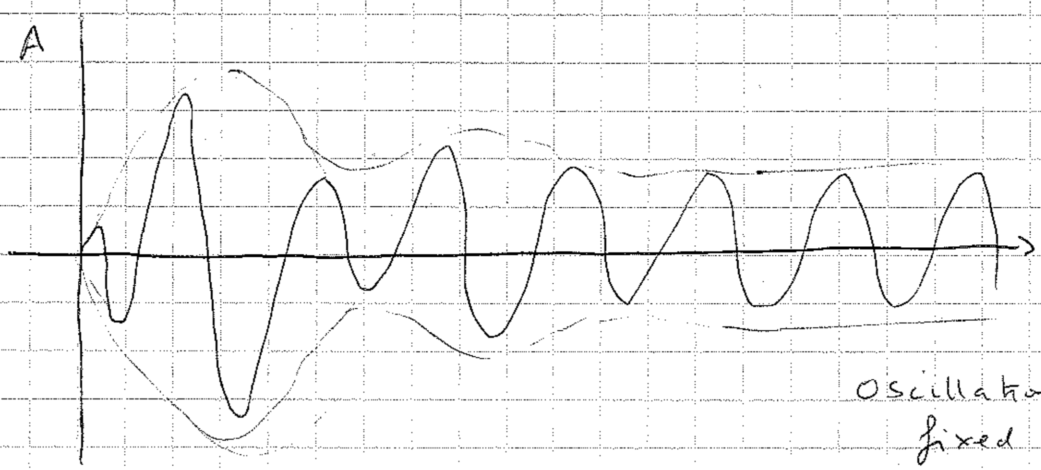
Therefore we postulate the following as a solution:

$$x_3 = B e^{-\gamma t/2} \cos(\omega_1 t + \beta) + A \cos(\omega t - \delta)$$

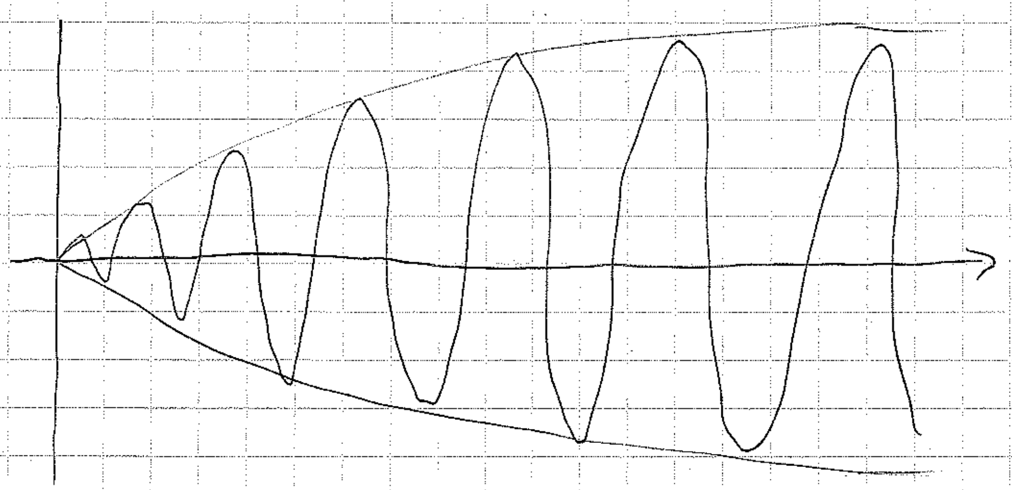
$$\text{with} \quad \omega_1 = \left(\omega_0^2 - \frac{\gamma^2}{4}\right)^{1/2}$$

while A & δ were given in last lecture.

Setting B & β would follow the same procedure as before - but it's "messier".



Oscillation reaches fixed amplitude as transient dies away



At resonance frequency amplitude simply grows to its max.