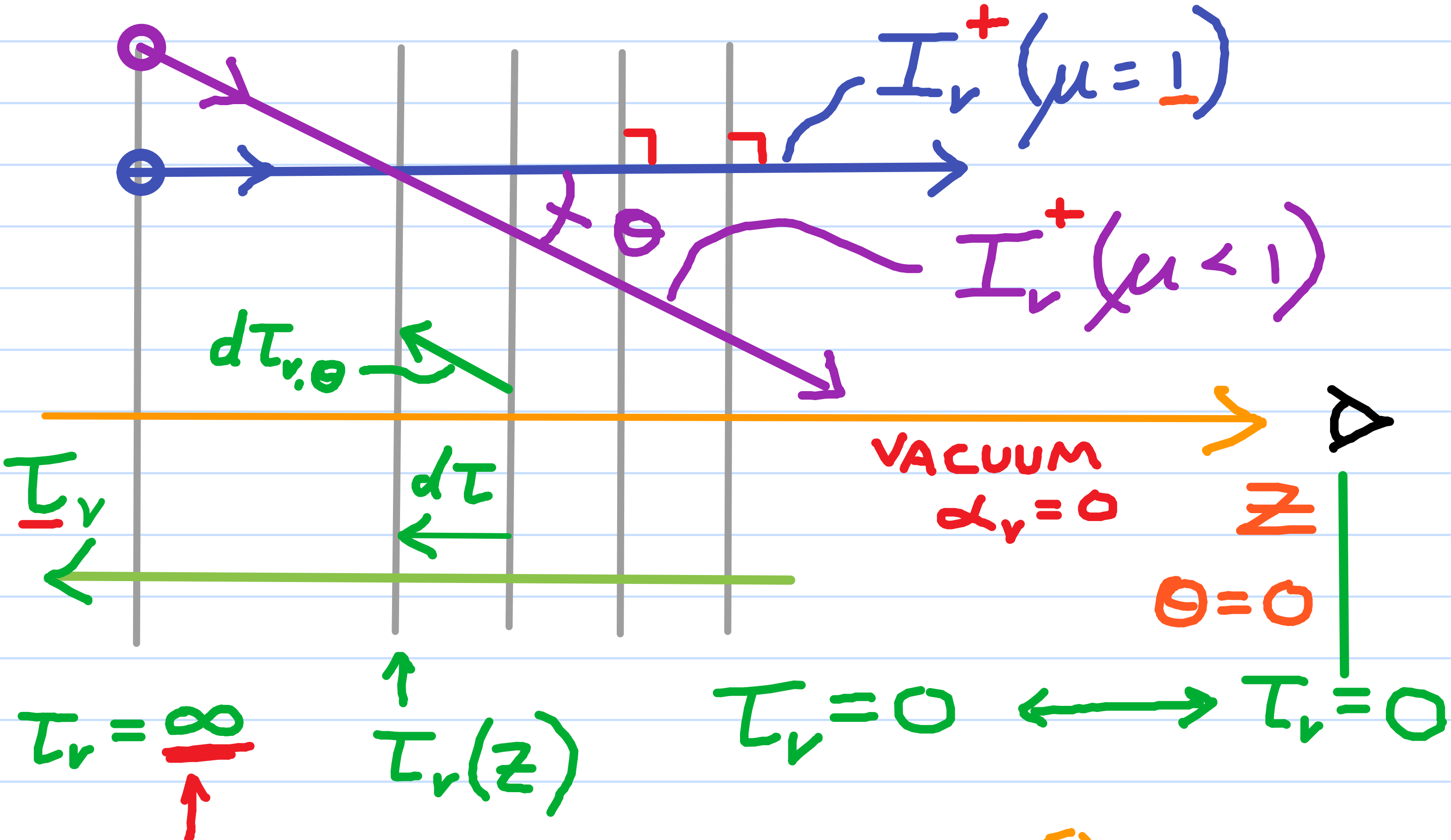
FORMAL SOLUTION TO RAD. TRANS. Eq. IN 1D MODEL:

1) OUTGOING BEAMS I_ν^+

$(1 \geq \mu > 0,\ t_\nu > \tau_\nu)$:



$$I^+(\tau_\nu, \mu) = I_\nu^+(\infty, \mu)\, \cancel{e^{-(\infty - \tau_\nu)/\mu}} \rightarrow 0$$

$$+ \int_{t_\nu = \tau_\nu}^{\infty} S_\nu(t_\nu)\, e^{-\overbrace{(t_\nu - \tau_\nu)}^{+ve}/\mu}\, \frac{dt_\nu}{\mu}$$

OUTGOING BEAMS I_ν^+

$(1 \geq \mu > 0,\ t_\nu > \tau_\nu)$:

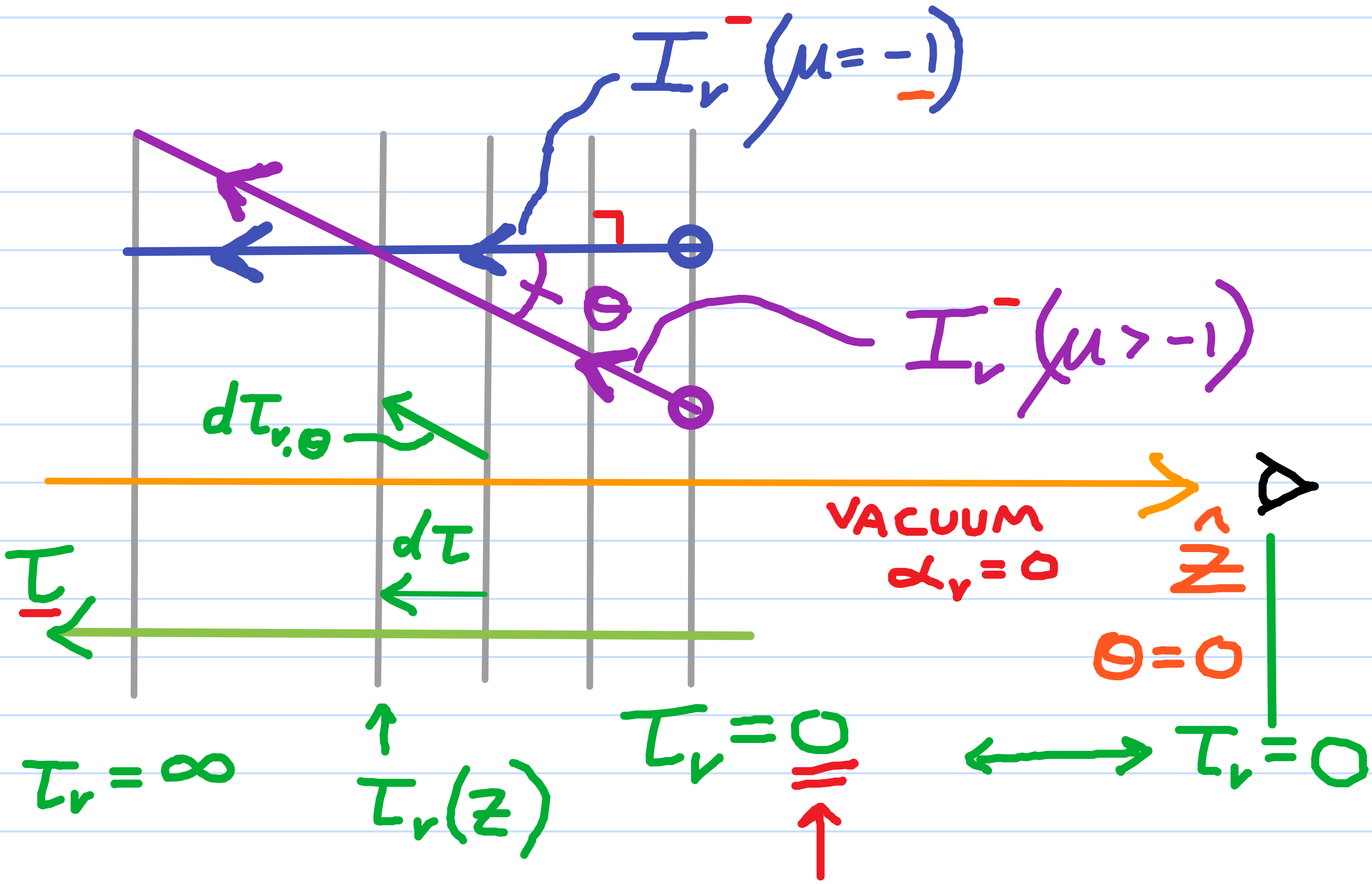
$$I^+(\tau_\nu, \mu) = \frac{1}{\mu} \int_{t_\nu=\tau_\nu}^{\infty} S_\nu(t_\nu)\, e^{-(t_\nu - \tau_\nu)/\mu}\, dt_\nu$$

THEN, SURFACE FLUX $(\tau_\nu = 0)$:

$$F_\nu^+(0) = \int_0^1 \mu\, I_\nu^+(0, \mu)\, d\mu$$

2) INCOMING BEAMS I_ν^-

$(-1 \le \mu < 0,\ t_\nu \le \tau_\nu)$:



$$I^-(\tau_\nu,\mu) = I_\nu^-(0,\mu)\, e^{-(0-\tau_\nu)/\mu} - \int_{t_\nu=0}^{\tau_\nu} S_\nu(t_\nu)\, e^{-(t_\nu-\tau_\nu)/\mu} \frac{dt_\nu}{\mu} \ge 0$$

0

$-$ve

$-$ve

$-$ve

INCOMING BEAMS I_ν^-
$(-1 < \underline{\mu} < 0,\ t_\nu \leq \tau_\nu)$:

$$I^-(\tau_\nu, \mu) = -\frac{1}{\mu}\int_{t_\nu=0}^{\tau_\nu} S_\nu(t_\nu)\, e^{-(t_\nu-\tau_\nu)/\mu}\, dt_\nu$$

$$\underline{> 0}$$

ANGLE-AVERAGES OF FORMAL SOL'n IN 1D AXI-SYMMETRIC ATMOSPHERE

(ON RADIAL τ_ν-SCALE)

RECALL:

$$\langle x \rangle = \frac{1}{2} \oint_{-1}^{1} x(\mu)\, d\mu$$

$$n^{th} \text{ MOMENT} \equiv \frac{1}{2} \oint_{-1}^{1} \mu^n I_\nu \, d\mu$$

USEFUL: J_ν, H_ν, K_ν DIRECTLY FROM S_ν

n^{th} ANGLE MOMENT:

$$\frac{1}{2}\int_{-1}^{1}\mu^n I_\nu(\tau_\nu,\mu)\,d\mu$$

$$= \frac{1}{2}\int_{0}^{1}\mu^n I_\nu^{+}(\tau_\nu,\mu)\,d\mu$$

$$+ \quad \frac{1}{2}\int_{-1}^{0}\mu^n I_\nu^{-}(\tau_\nu,\mu)\,d\mu$$

$$= \frac{1}{2}\int_{0}^{1}\mu^n \int_{\tau_\nu}^{\infty} S_\nu(t_\nu)\, e^{-(t_\nu-\tau_\nu)/\mu}\,\frac{dt_\nu}{\mu}\,d\mu$$

$$+ \frac{1}{2}\int_{-1}^{0}\mu^n \int_{0}^{\tau_\nu} S_\nu(t_\nu)\, e^{-(\tau_\nu-t_\nu)/\overbrace{(-\mu)}^{+ve}}\,\frac{dt_\nu}{\underbrace{(-\mu)}_{+ve}}\,d\mu$$

IF $S_\nu(\cancel{\mu}) = S_\nu$:

$$\frac{1}{2}\int_{-1}^{1} \mu^n I_\nu(\tau_\nu,\mu)\, d\mu$$

$$= \frac{1}{2}\int_{\tau_\nu}^{\infty} S_\nu(t_\nu) \int_{0}^{1} \mu^n e^{-(t_\nu-\tau_\nu)/\mu} \frac{d\mu}{\mu}\, dt_\nu$$

$$+ \quad \frac{1}{2}\int_{0}^{\tau_\nu} S_\nu(t_\nu) \int_{-1}^{0} \mu^n e^{-(\tau_\nu-t_\nu)/(-\mu)} \frac{d\mu}{(-\mu)}\, dt_\nu$$

SUBSTITUTION OF VARIABLES:

FOR $0 < \mu < 1$: $\quad w \equiv \frac{1}{\mu} = \sec\theta$:

$$\therefore d\mu = -\frac{dw}{w^2} = -\mu\frac{dw}{w}$$

$$\mu = 1 \rightarrow w = 1\ ;\ \mu = 0 \rightarrow w = \infty$$

FOR $-1 < \mu < 0$: $\quad w \equiv -\frac{1}{\mu} = -\sec\theta$:

$$\therefore d\mu = \mu\frac{dw}{w}$$

$$\mu^n = \frac{(-1)^n}{w^n}$$

$$\mu = -1 \rightarrow w = 1\ ;$$

$$\mu = 0 \rightarrow w = \lim_{\mu\to(-0)}\left(-\frac{1}{\mu}\right) = +\infty$$

$$\frac{1}{2}\int_{-1}^{1} \mu^n I_\nu(\tau_\nu, \mu)\, d\mu$$

$$= \frac{1}{2}\int_{\tau_\nu}^{\infty} S_\nu(t_\nu) \int_{w=\infty}^{1} \frac{e^{-(t_\nu-\tau_\nu)w}}{w^{n+1}}\, dw\, dt_\nu$$

$$+ \frac{(-1)^n}{2}\int_{0}^{\tau_\nu} S_\nu(t_\nu) \int_{w=1}^{\infty} \frac{e^{-(\tau_\nu-t_\nu)w}}{w^{n+1}}\, dw\, dt_\nu$$

-RUTTEN, p. 78

EXPONENTIAL INTEGRAL Fn, $E_m(x)$:

$$E_m(x) \equiv \int_{w=1}^{\infty} \frac{e^{-xw}}{w^m}\, dw \,,$$

$$x \geq 0 \,; \quad m = 1, 2, 3, \ldots$$

FOR $X \gg 1$:

$$E_m(x) \approx \frac{e^{-x}}{x}$$

n^{th} ANGLE-MOMENT : $m = n+1$:

$$\frac{1}{2}\int_{-1}^{1} \mu^n I_\nu(\tau_\nu,\mu)\,d\mu$$

$$= \frac{1}{2}\int_{\tau_\nu}^{\infty} S_\nu(t_\nu)\, E_{\underline{n+1}}\overset{>0}{(t_\nu-\tau_\nu)}\,dt_\nu$$

$$+ \frac{(-1)^n}{2}\int_{0}^{\tau_\nu} S_\nu(t_\nu)\, E_{\underline{n+1}}\overset{>0}{(\tau_\nu-t_\nu)}\,dt_\nu$$

ADVANTAGE :

$E_m(x)$ A WELL-KNOWN SPECIAL fn

Eq. J_ν : $n=0$ $(m=1)$

$$J_\nu(\tau_\nu) = \frac{1}{2}\int_{\tau_\nu}^{\infty} S_\nu(t_\nu)\, E_1(t_\nu - \tau_\nu)\, dt_\nu$$

$$+ \frac{1}{2}\int_{0}^{\tau_\nu} S_\nu(t_\nu)\, E_1(\tau_\nu - t_\nu)\, dt_\nu$$

$$\therefore\ J_\nu(\tau_\nu) = \frac{1}{2}\int_{0}^{\infty} S_\nu(t_\nu)\, E_1(|t_\nu - \tau_\nu|)\, dt_\nu$$

– SCHWARZSCHILD Eq.

J_ν IS AN EVEN ANGLE-MOMENT

Eq. $F_\nu = 4\pi H_\nu$: : $n=1$ $(m=2)$

$$F_\nu(\tau_\nu) = \left\{ 2\pi \int_{\tau_\nu}^{\infty} S_\nu(t_\nu)\, E_2(t_\nu - \tau_\nu)\, dt_\nu \right.$$

$$\left. - \, 2\pi \int_0^{\tau_\nu} S_\nu(t_\nu)\, E_2(\tau_\nu - t_\nu)\, dt_\nu \right\}$$

$$= F_\nu^{+}(\tau_\nu) - F_\nu^{-}(\tau_\nu)$$

– MILNE Eq.

F_ν IS AN ODD ANGLE-MOMENT

OVERVIEW OF I_ν AND ITS MOMENTS:

$$\begin{cases} I^{+}(\tau_\nu,\mu) = \dfrac{1}{\mu}\displaystyle\int_{t_\nu=\tau_\nu}^{\infty} S_\nu(t_\nu)\, e^{-(t_\nu-\tau_\nu)/\mu}\, dt_\nu \\[2ex] I^{-}(\tau_\nu,\mu) = -\dfrac{1}{\mu}\displaystyle\int_{t=0}^{\tau_\nu} S_\nu(t_\nu)\, e^{-(t_\nu-\tau_\nu)/\mu}\, dt_\nu \end{cases}$$

$$J_\nu(\tau_\nu) = \frac{1}{2}\int_0^\infty S_\nu(t_\nu)\, E_1(|t_\nu-\tau_\nu|)\, dt_\nu$$

$$F_\nu(\tau_\nu) = \left\{ 2\pi\int_{\tau_\nu}^{\infty} S_\nu(t_\nu)\, E_2(t_\nu-\tau_\nu)\, dt_\nu - 2\pi\int_0^{\tau_\nu} S_\nu(t_\nu)\, E_2(\tau_\nu-t_\nu)\, dt_\nu \right\}$$

$$\underline{\underline{K_\nu}}(\tau_\nu) = \frac{1}{2}\int_0^\infty S_\nu(t_\nu)\, E_3(|t_\nu-\tau_\nu|)\, dt_\nu = \frac{c}{4\pi}\underline{P_\nu}(\tau_\nu)$$

SURFACE VALUES: $(\tau_\nu = 0)$

$$I_\nu^+(0,\mu) = \frac{1}{\mu}\int_0^\infty S_\nu(t_\nu)\, e^{-t_\nu/\mu}\, dt_\nu$$

$$J_\nu(0) = \frac{1}{2}\int_0^\infty S_\nu(t_\nu)\, E_1(t_\nu)\, dt_\nu$$

$$\mathcal{F}_\nu(0) = 2\pi\int_0^\infty S_\nu(t_\nu)\, E_2(t_\nu)\, dt_\nu$$

OPERATOR FORMALISM:

MONOCHROMATIC Λ-OPERATOR:

$$\Lambda_{\tau}[f(\tau)] \equiv \frac{1}{2}\int_{t=0}^{\infty} f(t)\, E_1(|t-\tau|)\, dt$$

THEN: $J_\nu(\tau_\nu) = \Lambda_{\tau,\nu}[S_\nu(\tau_\nu)]$

-SCHWARZSCHILD Eq.

SIMILARLY: $F_\nu(\tau_\nu) = \Phi_{\tau,\nu}[S_\nu(\tau_\nu)]$

-MILNE Eq.

ADVANTAGE:

COMPACTNESS

OPERATOR THEORY IS POWERFUL

GRAPHICAL INTERPRETATION

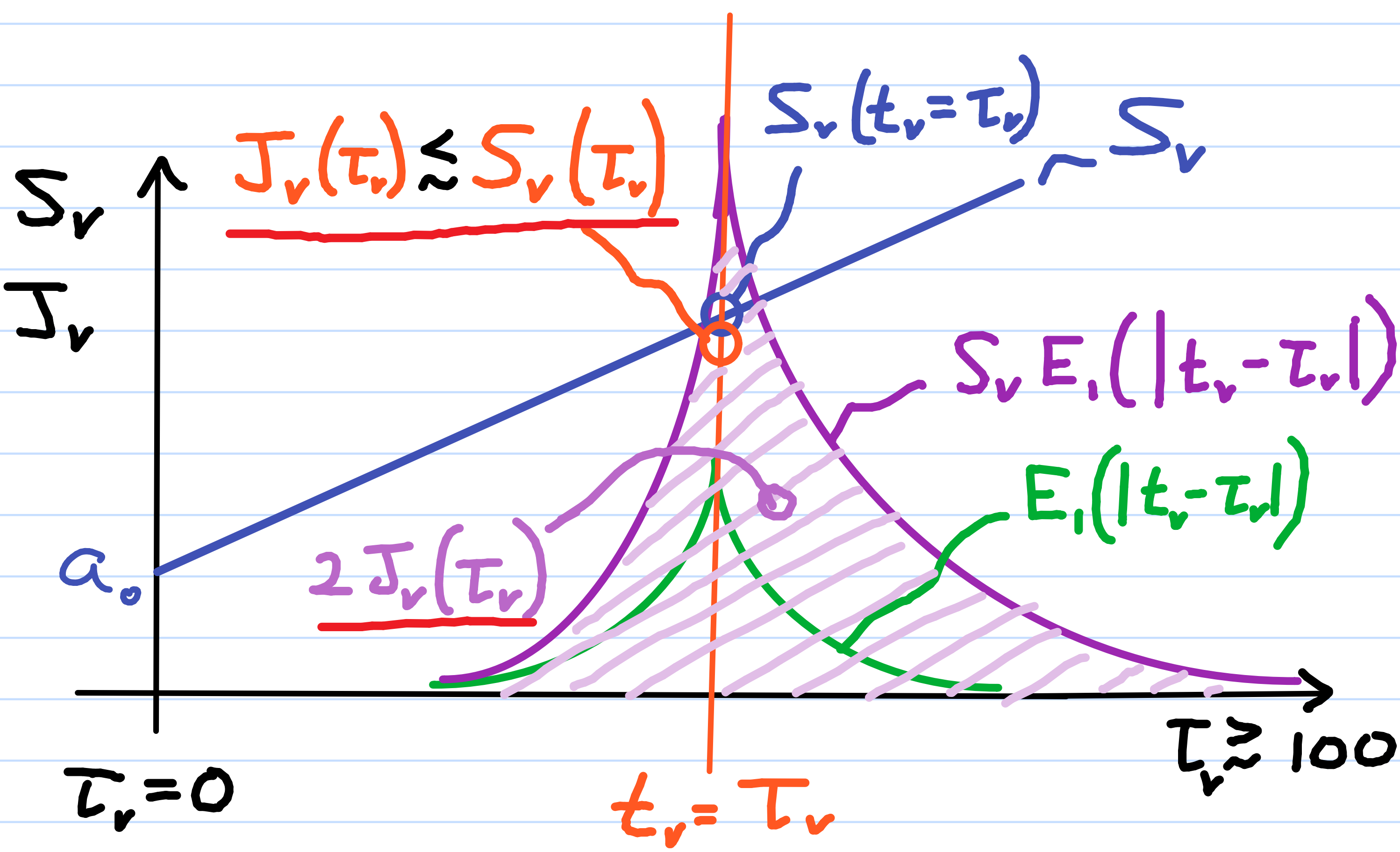
- RUTTEN, p. 80

SIMPLE MODEL:

$$S_\nu(\tau_\nu) = a_0 + a_1 \tau_\nu \qquad (a_0, a_1 > 0)$$

$$J_\nu(\tau_\nu) = \frac{1}{2}\int_0^\infty S_\nu(t_\nu)\, E_1(|t_\nu - \underline{\tau_\nu}|)\, dt_\nu \;:$$

$\underline{\tau_\nu > 1}$:

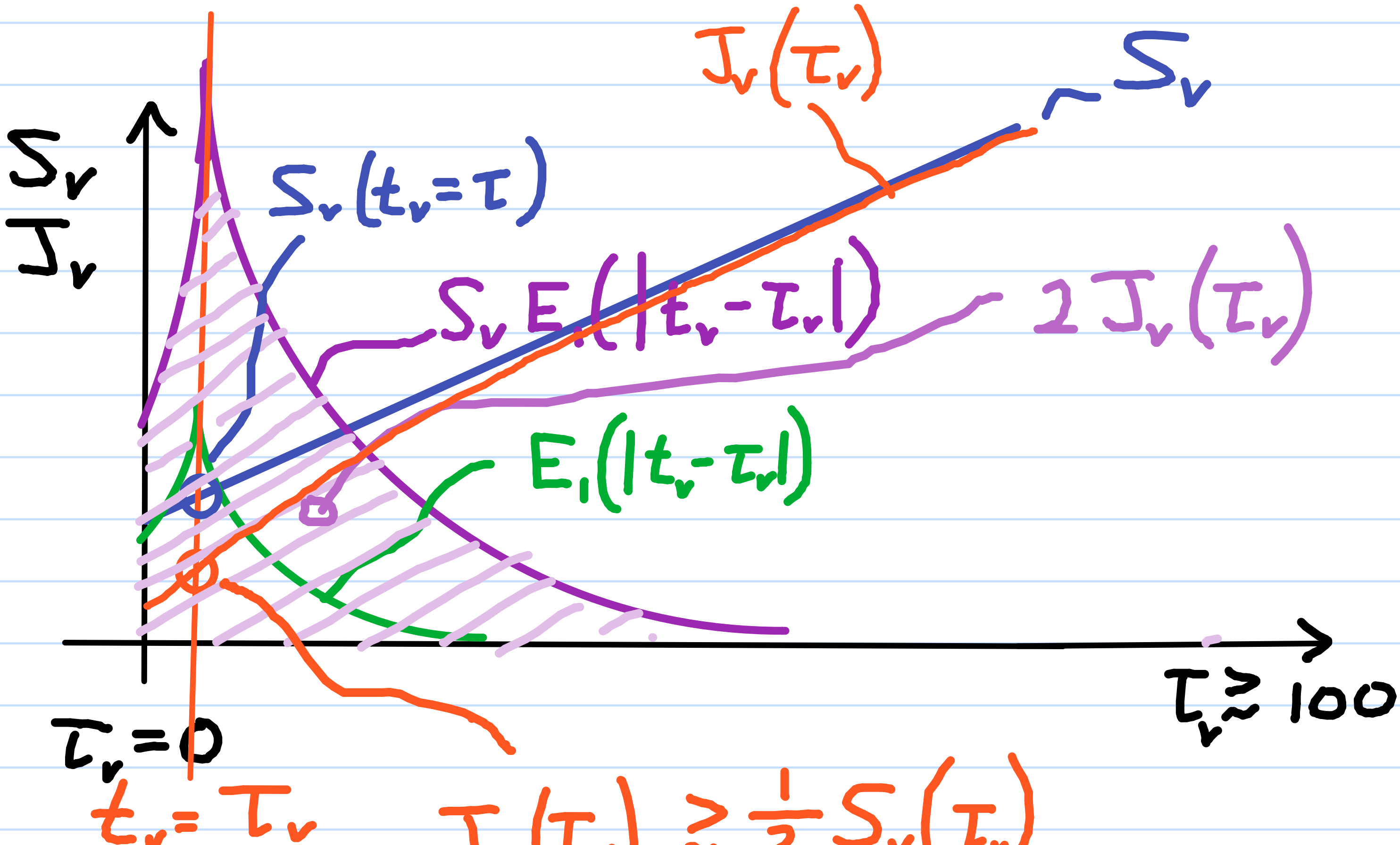


GRAPHICAL INTERPRETATION

-RUTTEN, p. 80

$$J_\nu(\tau_\nu) = \frac{1}{2}\int_0^\infty S_\nu(t_\nu)\, E_1(|t_\nu - \tau_\nu|)\, dt_\nu :$$

$\tau_\nu \gtrsim 0$:



$$J_\nu(\tau_\nu) \gtrsim \frac{1}{2} S_\nu(\tau_\nu)$$

$$\tilde{F}_\nu(\tau_\nu) = \left\{ 2\pi \int_{\tau_\nu}^{\infty} S_\nu(t_\nu)\, E_2(t_\nu - \tau_\nu)\, dt_\nu \right.$$

$$\left. - 2\pi \int_0^{\tau_\nu} S_\nu(t_\nu)\, E_2(\tau_\nu - t_\nu)\, dt_\nu \right\}$$

$\tau_\nu > 1$: