A Primer on
Magnetohydrodynamics

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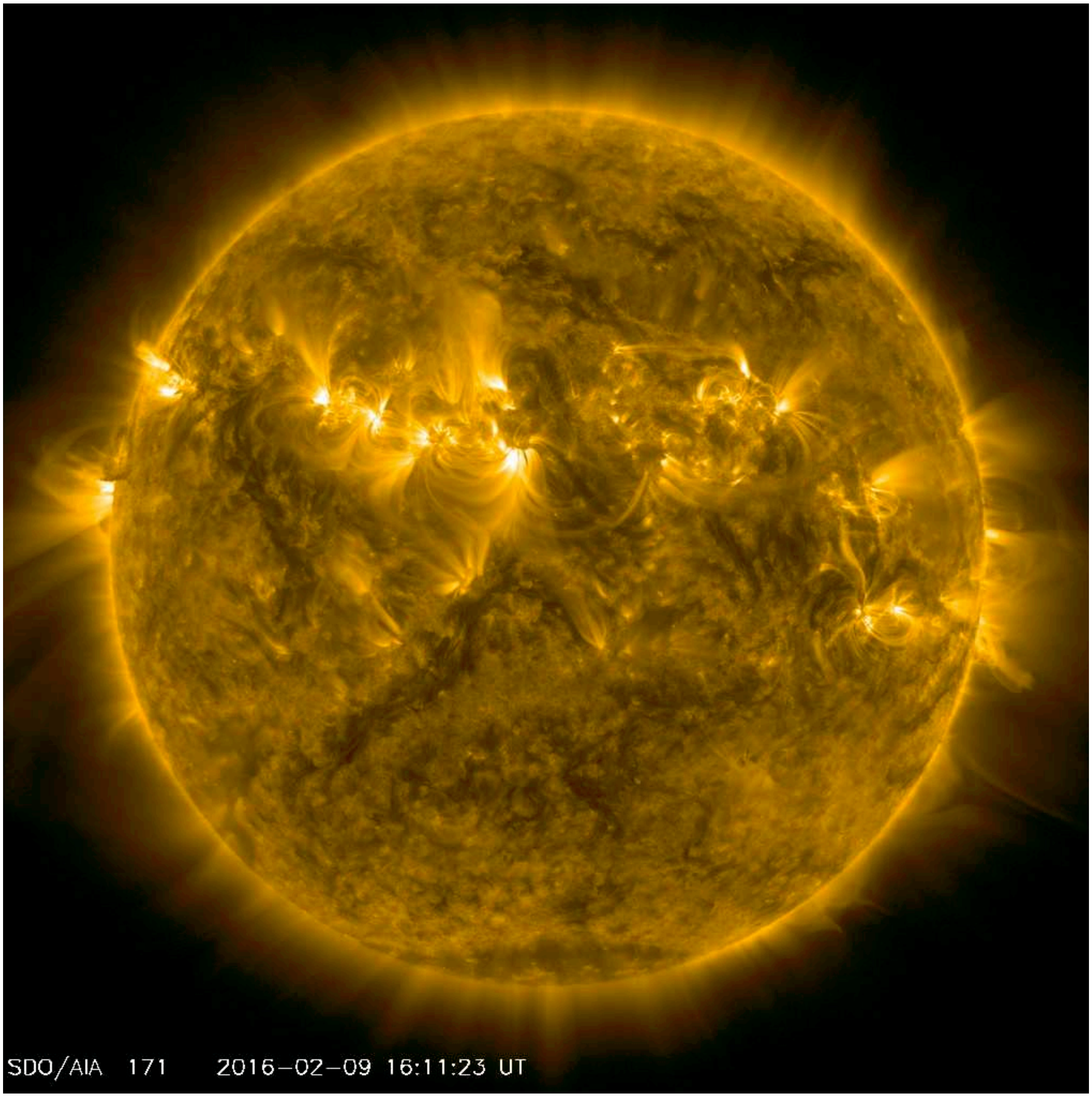
To those uninitiated to the magic of MHD, I can think of no better introduction to the subject than the following video, prepared by cinematographers engaged by NASA to highlight the work of the dedicated scientists, technicians, engineers, and staff who made the Solar Dynamic Observatory the spectacular success it was.

Prepare to have the hair on the back of your neck stand on end: SDO, live MHD



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Preface

This primer, along with the accompanying document *A Primer on ZEUS-3D*, were prepared for a mini-series of two lectures I gave at the University of Victoria in February, 2015. The intent of these two primers was to give an overview of MHD, both from a theoretical standpoint and as a practical computational tool, to (astro)physicists who may not have had any formal training in the subject, but who may nevertheless be contemplating problems in which MHD is a significant component.

I view these primers as the first salvo of what I'd like to develop into two mini-courses that would form the core of a summer "*ZEUS*-school" for theoretical and computational MHD, offered to users and potential users of *ZEUS-3D* and AZEuS worldwide. Time will tell if this idea ever gets off the ground!

In the meantime and inasmuch as these notes may help others, the reader is free to use, distribute, and modify them as needed so long as they remain in the public domain and are passed on to others free of charge.

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February, 2016

Primers by David Clarke:

1. A FORTRAN Primer
2. A UNIX Primer
3. A DBX (debugger) Primer
4. A Primer on Tensor Calculus
5. A Primer on Magnetohydrodynamics
6. A Primer on *ZEUS-3D*

I also give a link to David R. Wilkins' excellent primer Getting Started with LaTeX, in which I have added a few sections on adding figures, colour, and HTML links.

1 A bit of History

We have to learn again that science without contact with experiments is an enterprise which is likely to go completely astray into imaginary conjecture.

Hannes Alfvén, *father of MHD*

The branch of physics known as *magnetohydrodynamics* (MHD) was invented by the Swedish (astro)physicist/engineer, Hannes Olof Gösta Alfvén (1908–1995)[1] who, among other accomplishments, was the first (with N. Herlofson) in 1950 to detect and identify non-thermal (synchrotron) radiation from the cosmos, was the first in 1963 to predict the large-scale filamentary nature of the universe, and is universally regarded as the inventor and father of MHD for which he shared the 1970 Nobel prize in physics.

When his theory predicting the existence of what are now known as *Alfvén waves* was first published[2], it was received with considerable criticism and ridicule. This was the first time anyone ever suggested that electromagnetic waves of any sort could be supported by a conducting medium (and exactly opposite to what had been presumed "obvious" since Maxwell, namely that all conductors attenuate electromagnetic waves within a *skin depth*), and comments such as *'If such a thing were possible, Maxwell himself would have discovered it'* was often how Alfvén's ideas were dismissed. It is said that in the late 1950's while giving one of his lectures at the University of Chicago, none other than Enrico Fermi was heard by all to say "Of course!", at which point the ridicule stopped. However, Alfvén's real triumph happened in 1958 when Alfvén waves were indisputably detected in the lab. It was from that point in time when MHD suddenly became part of "mainstream physics", even to the point of being considered "obvious"!

MHD is an example of classical physics, and that it was 35 years *after* the advent of quantum mechanics before physicists started to become familiar with it is an historical anomaly. Despite its classical character, MHD is a relatively new branch of physics.

2 Conserved Quantities

MHD is based on the conservation of mass, momentum (Newton's second law), energy, and magnetic flux. Consider an ensemble of charged particles within some volume V, and let these particles interact with each other via elastic collisions. If their collective mass,

[1] http://public.lanl.gov/alp/plasma/people/alfven.html

[2] Alfvén, H., 1942, *Existence of electromagnetic-hydrodynamic waves*, Nature, v. 150, p. 405

momentum, total energy, and magnetic flux are M, $\vec{S}$, $E_{\rm T}$, and Φ respectively, then:

$$\frac{dM}{dt} = 0; \qquad \text{conservation of mass} \tag{1}$$

$$\frac{d\vec{S}}{dt} = \sum \vec{F}_{\rm ext}; \qquad \text{Newton's Second Law} \tag{2}$$

$$\frac{dE_{\rm T}}{dt} = \mathcal{P}_{\rm app}; \qquad \text{conservation of total energy} \tag{3}$$

$$\frac{d\Phi}{dt} = 0, \qquad \text{conservation of magnetic flux} \tag{4}$$

where $\vec{F}_{\rm ext}$ are all forces *external* to the ensemble of particles, while $\mathcal{P}_{\rm app}$ is the rate at which work is done (power) by all forces *applied* to the ensemble of particles.

Forces external to the ensemble of particles include gravity, Lorentz force, viscous drag and collisions from neighbouring (outside the volume V) particles. Since the total energy, $E_{\rm T}$, includes the gravitational and magnetic energy densities, the only external forces considered to be *applied* are the collisions.

A fluid is defined by how collisions among the particles are manifest. If a sufficient number of particles collide per unit time, the force of these collisions is isotropic and felt as a *pressure gradient.* Likewise, their charges will be distributed so that no external static electric field can be supported. In this case, we say that the distance between collisions (mean-free path), δl, is much smaller than the size of the box, Δl, of the volume V, which in turn is smaller than the smallest physical scale of interest, $\mathcal{L}$:

$$\delta l \ll \Delta l < \mathcal{L}$$

If this inequality is not valid, then the system of particles comes under the realm of *plasma physics* rather than MHD, and is governed by the so-called *Vlasov equation* (*Boltzmann equation* for neutral particles).

3 Theorem of Hydrodynamics

Definition: An *extensive quantity* is proportional to the amount of substance being considered. Examples include the mass, energy, volume, *etc.*

Definition: An *intensive quantity* is independent of the amount of substance being considered. Examples include the density, pressure, temperature, *etc.*

For every extensive quantity, $Q(V,t)$, of a sample, a corresponding intensive quantity, $q(\vec{r},t)$, may be given by:

$$q(\vec{r},t) = \lim_{\Delta V \to 0} \frac{\Delta Q(V,t)}{\Delta V} = \frac{\partial Q(V,t)}{\partial V}. \tag{5}$$

Conversely,

$$Q(V,t) = \int_V q(\vec{r},t)\, dV. \tag{6}$$

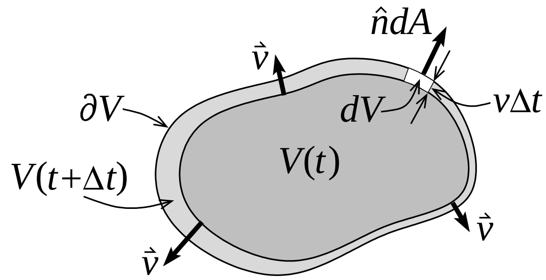

Figure 1: A volume, $V(t)$, expands to volume $V(t+\Delta t)$ in a time Δt by adding to $V(t)$ a "skin" of surface area ∂V and thickness $v\Delta t$.

Theorem of Hydrodynamics.[3] *If the time dependence of an extensive variable, Q, is given by:*

$$\frac{dQ}{dt} \;=\; \Sigma, \tag{7}$$

where Σ represents (time-dependent) "source terms", then the corresponding intensive quantity, $q(\vec{r},t)$, is governed by:

$$\frac{\partial q}{\partial t} + \nabla\cdot(q\,\vec{v}) \;=\; \sigma, \tag{8}$$

where $\vec{v} = d\vec{r}/dt$, $Q = \int_V q\,dV$, $\Sigma = \int_V \sigma\,dV$, and where the product $q\,\vec{v}$ must be a differentiable *function of the coordinates*[4].

Proof:

$$\frac{dQ}{dt} = \Sigma \quad\Rightarrow\quad \frac{d}{dt}\int_V q dV \;=\; \int_V \sigma dV,$$

where, in general, the volume element $V = V(t)$ also varies in time. Thus,

$$\begin{aligned}
\frac{d}{dt}\int_{V(t)} q dV \;&=\; \lim_{\Delta t\to 0}\frac{1}{\Delta t}\left[\int_{V(t+\Delta t)} q(\vec{r},t{+}\Delta t)dV - \int_{V(t)} q(\vec{r},t)dV\right]\\
&=\; \lim_{\Delta t\to 0}\frac{1}{\Delta t}\left[\int_{V(t+\Delta t)-V(t)} q(\vec{r},t{+}\Delta t)dV + \int_{V(t)} q(\vec{r},t{+}\Delta t)dV - \int_{V(t)} q(\vec{r},t)dV\right]\\
&=\; \lim_{\Delta t\to 0}\frac{1}{\Delta t}\int_{\Delta V} q(\vec{r},t{+}\Delta t)dV + \lim_{\Delta t\to 0}\frac{1}{\Delta t}\int_{V(t)}\big[q(\vec{r},t{+}\Delta t) - q(\vec{r},t)\big]dV
\end{aligned}$$

where, as shown in Fig. 1, the difference in volume, $\Delta V = V(t+\Delta t) - V(t)$, is the surface integral over the closed surface, ∂V, of the "skin thickness", $\vec{v}\Delta t$, times the area differential,

[3]This theorem is a variant of *Reynolds' transport theorem*.

[4]Requiring a function to be integrable is less severe than requiring it to be differentiable. The former requires only that there be no singularities of first order or greater in the function [*i.e.*, if $f(x{-}x_0) \sim (x{-}x_0)^p$ near x_0, $p > -1$ for the function to be integrable through x_0], whereas the latter also requires the function to be continuous at x_0 (and for $p \geq 0$).

$\hat{n}\Delta A$. Thus, within ΔV, the volume differential is $dV = (\vec{v}\Delta t)\cdot(\hat{n}dA)$, and we have:

$$\begin{aligned}
\frac{d}{dt}\int_{V(t)} q dV \;&=\; \lim_{\Delta t\to 0}\frac{1}{\cancel{\Delta t}}\oint_{\partial V} q(\vec{r}, t{+}\Delta t)(\vec{v}\cancel{\Delta t})\cdot(\hat{n}dA) + \int_{V(t)}\lim_{\Delta t\to 0}\frac{q(\vec{r}, t{+}\Delta t) - q(\vec{r}, t)}{\Delta t} dV \\
&=\; \oint_{\partial V} q(\vec{r}, t)\vec{v}\cdot\hat{n}dA + \int_{V(t)}\frac{\partial q(\vec{r}, t)}{\partial t} dV \\
&=\; \int_{V(t)} \nabla\cdot\big(q(\vec{r}, t)\vec{v}\big)dV + \int_{V(t)}\frac{\partial q(\vec{r}, t)}{\partial t} dV \quad \textit{(using Gauss' theorem)} \\
&=\; \int_{V(t)}\bigg(\frac{\partial q(\vec{r}, t)}{\partial t} + \nabla\cdot\big(q(\vec{r}, t)\vec{v}\big)\bigg)dV \;=\; \int_{V(t)} \sigma(\vec{r}, t) dV.
\end{aligned}$$

$$\Rightarrow \qquad \int_V \bigg(\frac{\partial q}{\partial t} + \nabla\cdot(q\,\vec{v}) - \sigma\bigg)dV \;=\; 0.$$

As this is true for any volume, V, the integrand must be zero, proving the theorem. □

4 Equations of Hydrodynamics

1. Continuity equation: Let $Q = M$, mass of fluid sample. Then $q = \rho$ (mass density) is the corresponding intensive variable and, from equation (1), $\Sigma = 0 \Rightarrow \sigma = 0$. Thus, the theorem of HD requires:

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\,\vec{v}) \;=\; 0.} \tag{9}$$

2. Momentum equation: Let $Q = \vec{S}$, total momentum of fluid sample. Then, $q = \vec{s} = \rho\vec{v}$ (momentum density) is the corresponding intensive variable and from equation (2), $\Sigma = \sum \vec{F}_{\rm ext} \Rightarrow \sigma = \sum \vec{f}_{\rm ext}$, the *external force densities*. Thus, the theorem of HD requires:

$$\frac{\partial \vec{s}}{\partial t} + \nabla\cdot(\vec{s}\,\vec{v}) \;=\; \sum \vec{f}_{\rm ext}, \tag{10}$$

where the Cartesian representation of the divergence term is:

$$\nabla\cdot(\vec{s}\,\vec{v}) \;=\; \Big(\nabla\cdot(s_x\vec{v}), \nabla\cdot(s_y\vec{v}), \nabla\cdot(s_z\vec{v})\Big).$$

For ideal HD, external force densities include terms arising from pressure gradients and gravity. For pressure gradients, consider a small cube with edge length Δx and face area ΔA (Fig. 2a). If the pressure at the left and right sides of the cube are respectively $p(x)$ and $p(x + \Delta x)$, then the net pressure force acting on the cube in the x-direction is given by:

$$F(x+\Delta x) + F(x) \;=\; -p(x+\Delta x)\,\Delta A + p(x)\,\Delta A \;=\; -\frac{\Delta p}{\Delta x}\,\Delta A\,\Delta x \;=\; -\frac{\Delta p}{\Delta x}\,\Delta V.$$

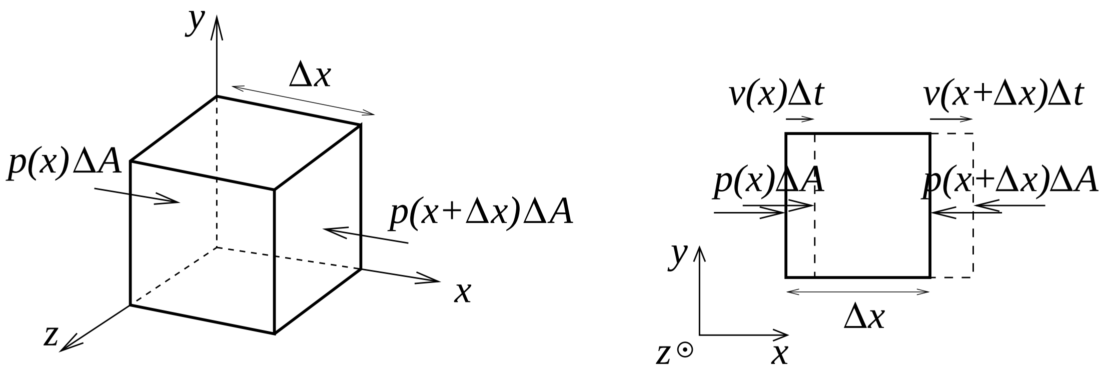

Figure 2: a) (left) A cube of edge length Δx with external pressure forces acting on the x-faces indicated. b) (right) An x-y cut through the cube on the left showing both the pressure forces on and motion of the x-faces.

Thus, the pressure force density in the x-direction is:

$$f_x = \frac{\sum F_x}{\Delta V} = -\frac{\Delta p}{\Delta x} \rightarrow -\frac{\partial p}{\partial x} \quad \text{as } \Delta x \rightarrow 0.$$

Accounting for all three components, we have for the pressure force density:

$$\vec{f}_p = -\nabla p. \tag{11}$$

As for gravity, if the mass of the fluid sample is ΔM, then the gravitational force on the mass is $-\Delta M\,\nabla\phi$, where ϕ is the local gravitational potential. Thus, the gravitational force density is:

$$\vec{f}_\phi = -\frac{\Delta M\,\nabla\phi}{\Delta V} \rightarrow -\rho\,\nabla\phi \quad \text{as } \Delta V \rightarrow 0. \tag{12}$$

Substituting both equations (11) and (12) into (10) yields the *momentum equation*:

$$\boxed{\frac{\partial \vec{s}}{\partial t} + \nabla\cdot(\vec{s}\,\vec{v}) = -\nabla p - \rho\,\nabla\phi.} \tag{13}$$

3. Energy equation: Let $Q = E_{\rm T}$ be the total energy of the fluid sample:

$$E_{\rm T} = \frac{1}{2}(\Delta M)v^2 + E + (\Delta M)\phi,$$

where E is the internal energy. The corresponding intensive variable is the total energy density,

$$e_{\rm T} = \frac{1}{2}\rho v^2 + e + \rho\phi, \tag{14}$$

where e is the internal energy density, related to the pressure by the ideal gas law:

$$p = (\gamma - 1)e, \tag{15}$$

where γ is the usual ratio of specific heats.

From equation (3), $\Sigma = \mathcal{P}_{\rm app} \Rightarrow \sigma = p_{\rm app}$, the rate at which work is done on a unit volume of fluid by all *applied* forces. Thus, the theorem of HD requires:

$$\frac{\partial e_{\rm T}}{\partial t} + \nabla \cdot (e_{\rm T}\vec{v}) \; = \; p_{\rm app}. \tag{16}$$

The applied power is the rate at which work is done by the external fluid on the fluid sample (collisions, and thus pressure) as the latter expands or contracts within the former. Thus, consider a small cubic sample of fluid with dimension Δx in the x-direction and cross sectional area ΔA (Fig. 2). The pressure force exerted on the left face of the cube is $F(x) = +p(x)\Delta A$ and, in time Δt, the left face is displaced by $v_x(x)\Delta t$. Thus, the work done on the left face by the external fluid is $\Delta W_{\rm L} = +p(x)\, v_x(x)\Delta t\, \Delta A$. Similarly, the work done on the right face is $\Delta W_{\rm R} = -p(x+\Delta x)\, v_x(x+\Delta x)\Delta t\, \Delta A$, and the net work done on the fluid cube by the external medium is:

$$\Delta W \; = \; \Delta W_{\rm L} + \Delta W_{\rm R} \; = \; p(x)\, v_x(x)\Delta t\, \Delta A - p(x+\Delta x)\, v_x(x+\Delta x)\Delta t\, \Delta A$$

$$\Rightarrow \quad \mathcal{P}_{\rm app} \; = \; \frac{\Delta W}{\Delta t} \; = \; -\,\Delta A\, \Delta x\, \frac{p(x+\Delta x)\, v_x(x+\Delta x) - p(x)\, v_x(x)}{\Delta x} \; = \; -\,\Delta V\, \frac{\Delta(pv_x)}{\Delta x}.$$

Thus, the *applied power density* is given by:

$$p_{\rm app} \; = \; \frac{\mathcal{P}_{\rm app}}{\Delta V} \; = \; -\,\frac{\Delta(pv_x)}{\Delta x} \; \rightarrow \; -\frac{\partial(pv_x)}{\partial x}.$$

Taking into account similar terms in the y- and z-directions,

$$p_{\rm app} \; = \; -\nabla \cdot (p\,\vec{v}). \tag{17}$$

Substituting equation (17) into (16) yields:

$$\boxed{\frac{\partial e_{\rm T}}{\partial t} + \nabla \cdot [(e_{\rm T} + p)\,\vec{v}] \; = \; 0.} \tag{18}$$

5 The Induction Equation

In an electromagnetic field, a particle with charge q feels an electromagnetic force:

$$\vec{F}_{\rm EM} \; = \; q(\vec{E} + \vec{v} \times \vec{B}),$$

where $q\vec{E}$ is the *Coulomb* force and $q\vec{v} \times \vec{B}$ is the *Lorentz* force.

As $\vec{F}_{\rm EM}$ accelerates the charge, collisions with other particles increase and a drag force, much like that exerted on objects falling through the air, is exerted on the particle. This drag force is proportional to the charge on the particle, q, its velocity through the medium, $\vec{v}$, and the charge density of the medium, ρ_q:

$$\vec{F}_{\rm R} \; = \; -\eta q \vec{v} \rho_q,$$

where $-$ indicates $\vec{F}_{\rm R}$ points in the opposite direction as $\vec{v}$, and where η is the *resistivity* (proportionality constant).

Any particle acted upon by a drag force eventually attains a *terminal velocity* where the drag and driving forces balance. Thus, we have:

$$\vec{F}_{\rm EM} + \vec{F}_{\rm R} = q(\vec{E} + \vec{v}\times\vec{B} - \eta\rho_q\vec{v}) = 0 \quad \Rightarrow \quad \vec{E} + \vec{v}\times\vec{B} = \eta\vec{J}. \tag{19}$$

where $\vec{J} = \rho_q\vec{v}$ is the *current density*. Equation (19) is the generalised Ohm's Law in which, for a fluid, $\vec{v}$ is interpreted as the velocity of a charged fluid element rather than an individual charged particle.

Taking the curl of equation (19), we get:

$$\nabla\times\vec{E} + \nabla\times(\vec{v}\times\vec{B} - \eta\vec{J}) = 0 \quad \Rightarrow \quad \frac{\partial\vec{B}}{\partial t} = \nabla\times(\vec{v}\times\vec{B} - \eta\vec{J}),$$

the combined Faraday-Ohm law, where Faraday's Law ($\nabla\times E = -\partial\vec{B}/\partial t$) has been invoked. In the limit of an *ideal* fluid (where both viscosity and resistivity are zero), we get the *induction equation*:

$$\boxed{\frac{\partial\vec{B}}{\partial t} = \nabla\times(\vec{v}\times\vec{B}).} \tag{20}$$

Finally, from the Ampère-Maxwell Law:

$$\nabla\times\vec{B} = \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t} + \mu_0\vec{J} \quad \Rightarrow \quad \vec{J} = \rho_q\vec{v} = \frac{1}{\mu_0}\nabla\times\vec{B}, \tag{21}$$

if we ignore the displacement current on the grounds that the factor $1/c^2$ makes it negligible for non-relativistic fluid motions.

On comparing Faraday's Law with equation (20), we see that there is an *induced* electric field in the fluid given by:

$$\vec{E}_{\rm ind} = -\vec{v}\times\vec{B}. \tag{22}$$

It was general ignorance of this term that led many physicists of his day to dismiss Alfvén's theory. The idea that conducting media cannot support electromagnetic fields is predicated on the fact that free charges redistribute themselves nearly instantaneously (*e.g.*, within a skin depth) to negate any *static* electric field (whose field lines begin and end on free charges) that may try to establish itself. But the electric field in equation (22) is an *induced* field which forms closed loops. It is manifestly non-conservative and cannot be eliminated by the rearrangement of free charges. Thus, an MHD fluid is a conducting medium that can and does support electromagnetic fields and is why the physics of MHD is so rich.

6 Conservative and Primitive Forms of MHD

A magnetic field in the presence of a charged MHD fluid introduces another force density to the momentum equation. A parcel of fluid of charge q moving at velocity $\vec{v}$ in a background magnetic induction $\vec{B}$ feels a *Lorentz force density* of:

$$\vec{f}_{\rm L} = \rho_q\vec{v}\times\vec{B} = \vec{J}\times\vec{B} = \frac{1}{\mu_0}(\nabla\times\vec{B})\times\vec{B},$$

using equation (21). Thus, equation (13) becomes:

$$\boxed{\frac{\partial \vec{s}}{\partial t} + \nabla \cdot (\vec{s}\,\vec{v}) \;=\; -\nabla p - \rho\,\nabla\phi + \frac{1}{\mu_0}(\nabla \times \vec{B}) \times \vec{B}.} \tag{23}$$

The magnetic energy density associated with a magnetic field, $\vec{B}$ is given by:

$$e_B \;=\; \frac{B^2}{2\mu_0},$$

and thus the total energy density of an MHD fluid is:

$$e_{\rm T}^* \;=\; e_{\rm T} + e_B \;=\; \frac{1}{2}\rho v^2 + e + \frac{B^2}{2\mu_0} + \rho\phi.$$

It is left as an exercise to show that the evolution equation for $e_{\rm T}^*$ is therefore:

$$\boxed{\frac{\partial e_{\rm T}^*}{\partial t} + \nabla \cdot \left((e_{\rm T} + p)\vec{v} - \frac{1}{\mu_0}(\vec{v} \times \vec{B}) \times \vec{B}\right) \;=\; 0.} \tag{24}$$

The magnetic term in the divergence has a very simple physical interpretation. From equation 22, we have:

$$-\frac{1}{\mu_0}(\vec{v} \times \vec{B}) \times \vec{B} \;=\; \frac{1}{\mu_0}\vec{E}_{\rm ind} \times \vec{B} \;=\; \vec{S}_{\rm M},$$

the *Poynting vector*. Since the divergence of a Poynting vector is an energy flux, the last term in equation (24) represents energy transported along magnetic field lines from one parcel of fluid to another.

Thus, the equations of MHD are:

Equation set 1:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho\,\vec{v}) \;=\; 0; \tag{9}$$

$$\frac{\partial \vec{s}}{\partial t} + \nabla \cdot (\vec{s}\,\vec{v}) \;=\; -\nabla p - \rho\,\nabla\phi + \frac{1}{\mu_0}(\nabla \times \vec{B}) \times \vec{B}; \tag{23}$$

$$\frac{\partial e_{\rm T}^*}{\partial t} + \nabla \cdot (e_{\rm T}\vec{v}) \;=\; -\nabla \cdot \left(p\,\vec{v} - \frac{1}{\mu_0}\,(\vec{v} \times \vec{B}) \times \vec{B}\right) \tag{24}$$

$$\frac{\partial \vec{B}}{\partial t} - \nabla \times (\vec{v} \times \vec{B}) \;=\; 0, \tag{20}$$

where:

$$\vec{s} \;=\; \rho\vec{v}; \qquad e_{\rm T} = \frac{1}{2}\rho v^2 + \rho\phi; \qquad e_{\rm T}^* \;=\; e_{\rm T} + \frac{B^2}{2\mu_0}; \qquad p \;=\; (\gamma - 1)e,$$

are the constitutive equations. Equation set 1 are in the so-called *conservative form* because the time derivatives are of the intensive form of the conserved extensive variables, namely the mass, momentum, total energy, and magnetic flux.

There are numerous alternate forms of these equations, however, and it is left as another exercise to derive the following equations in the so-called *primitive form*:

Equation set 2:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \, \vec{v}) \; = \; 0; \tag{9}$$

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \, \vec{v} \; = \; -\frac{1}{\rho} \nabla p - \nabla \phi + \frac{1}{\rho \mu_0} (\nabla \times \vec{B}) \times \vec{B}; \tag{25}$$

$$\frac{\partial p}{\partial t} + \vec{v} \cdot \nabla p \; = \; -\gamma p \nabla \cdot \vec{v} \tag{26}$$

$$\frac{\partial \vec{B}}{\partial t} + \nabla \cdot (\vec{v}\vec{B} - \vec{B}\vec{v}) \; = \; 0, \tag{27}$$

where the induction equation (27) has been written in an alternative form using the dyadic products of $\vec{v}$ and $\vec{B}$ to make it look like a divergence rather than a curl. Note that equation set 2 is closed, and no constitutive equations are required. The variables $(\rho, \vec{v}, p, B)$ are known in MHD as the *primitive variables*, whence the designation of this equation set. Finally, equation (25) is known as *Euler's equation* after the most prolific mathematician of all time credited for being the first to write it down (in its HD form).

7 Fast, Alfvén, and slow waves

Consider the MHD equations in one Cartesian coordinate (x). Thus, $\partial_y = \partial_z = 0$, and equations (9), (25), (26), and (20) become:

Equation Set 3:

$$\partial_t \rho + v_x \, \partial_x \rho + \rho \, \partial_x v_x \; = \; 0; \tag{28}$$

$$\partial_t v_x + v_x \, \partial_x v_x + \frac{1}{\rho} \, \partial_x p + \frac{B_y}{\rho} \, \partial_x B_y + \frac{B_z}{\rho} \, \partial_x B_z \; = \; 0; \tag{29}$$

$$\partial_t v_y + v_x \, \partial_x v_y - \frac{B_x}{\rho} \, \partial_x B_y \; = \; 0; \tag{30}$$

$$\partial_t v_z + v_x \, \partial_x v_z - \frac{B_x}{\rho} \, \partial_x B_z \; = \; 0; \tag{31}$$

$$\partial_t p + v_x \, \partial_x p + \gamma p \, \partial_x v_x \; = \; 0; \tag{32}$$

$$\partial_t B_y + v_x \, \partial_x B_y + B_y \, \partial_x v_x - B_x \, \partial_x v_y \; = \; 0; \tag{33}$$

$$\partial_t B_z + v_x \, \partial_x B_z + B_z \, \partial_x v_x - B_x \, \partial_x v_z \; = \; 0. \tag{34}$$

Note that the x-component of the induction equation requires that:

$$\partial_t B_x \; = \; \partial_y (v_x B_y - v_y B_x) - \partial_z (v_z B_x - v_x B_z) \; = \; 0,$$

and B_x is constant in time, while the solenoidal condition requires that:

$$\nabla \cdot \vec{B} = \partial_x B_x = 0,$$

and B_x is also constant in space. Thus, B_x is constant everywhere for all time and we need only concern ourselves with the y- and z-components of the induction equation.

Equation set 3 may be written more compactly as:

$$\boxed{\partial_t|q\rangle + \mathsf{J}\,\partial_x|q\rangle = 0,} \tag{35}$$

where the ket of primitive variables, $|q\rangle$, and the Jacobian matrix for the primitive variables, J, are given by:

$$|q\rangle = \begin{bmatrix} \rho \\ v_x \\ v_y \\ v_z \\ B_y \\ B_z \\ p \end{bmatrix}; \qquad \mathsf{J} = \begin{bmatrix} v_x & \rho & 0 & 0 & 0 & 0 & 0 \\ 0 & v_x & 0 & 0 & B_y/\rho & B_z/\rho & 1/\rho \\ 0 & 0 & v_x & 0 & -B_x/\rho & 0 & 0 \\ 0 & 0 & 0 & v_x & 0 & -B_x/\rho & 0 \\ 0 & B_y & -B_x & 0 & v_x & 0 & 0 \\ 0 & B_z & 0 & -B_x & 0 & v_x & 0 \\ 0 & \gamma p & 0 & 0 & 0 & 0 & v_x \end{bmatrix}.$$

Equation (35) is a wave equation and we seek its *normal mode* solutions of the form:

$$|q(x,t)\rangle = |f(\xi)\rangle,$$

where $\xi = x - ut$, and where u is the speed of a particular normal mode (wave speed). Thus,

$$\partial_t|q\rangle = \frac{\partial\xi}{\partial t}\frac{d|f\rangle}{d\xi} = -u|f'\rangle; \qquad \partial_x|q\rangle = \frac{\partial\xi}{\partial x}\frac{d|f\rangle}{d\xi} = |f'\rangle. \tag{36}$$

Substituting equations (36) into (35), we get:

$$\mathsf{J}\,|f'\rangle = u|f'\rangle,$$

and u are the *eigenvalues* of the Jacobian J, while $|f'\rangle$ are (proportional to) its *eigenvectors*. In this primer, we shall look at the eigenvalues only.

To obtain the eigenvalues, we solve the linear algebra equation: $\det(\mathsf{J} - u\mathsf{I}) = 0$ which, after rather lengthy though straight-forward algebra, yields:

$$\begin{aligned} \det(\mathsf{J} - u\mathsf{I}) = {} & [u - (v_x - a_\mathrm{f})][u - (v_x - a_x)][u - (v_x - a_\mathrm{s})][u - v_x] \times \\ & [u - (v_x + a_\mathrm{s})][u - (v_x + a_x)][u - (v_x + a_\mathrm{f})] = 0, \end{aligned} \tag{37}$$

where a_f, a_x, and a_s are respectively the *fast magnetosonic speed*, the x-component of the *Alfvén speed*, and the *slow magnetosonic speed*. These are given by:

$$a_\mathrm{f} = \sqrt{\frac{1}{2}\left(a^2 + c_\mathrm{s}^2 + \sqrt{(a^2 + c_\mathrm{s}^2)^2 - 4a_x^2 c_\mathrm{s}^2}\right)}; \tag{38}$$

$$a_x = \frac{B_x}{\sqrt{\rho}}; \tag{39}$$

$$a_{\rm s} = \sqrt{\frac{1}{2}\left(a^2 + c_{\rm s}^2 - \sqrt{(a^2+c_{\rm s}^2)^2 - 4a_x^2 c_{\rm s}^2}\right)}, \tag{40}$$

where:

$$a^2 = \frac{B^2}{\rho} = \frac{B_x^2 + B_y^2 + B_z^2}{\rho}; \qquad c_{\rm s}^2 = \frac{\gamma p}{\rho},$$

are the squares of the Alfvén and sound speeds respectively. Therefore, the 1-D MHD equations admit *seven* characteristic (wave) speeds, namely:

$$\begin{array}{llll} u_1 = v_x - a_{\rm f}; & u_2 = v_x - a_x; & u_3 = v_x - a_{\rm s}; & u_4 = v_x; \\ u_5 = v_x + a_{\rm s}; & u_6 = v_x + a_x; & u_7 = v_x + a_{\rm f}, & \end{array} \tag{41}$$

which correspond to four distinct types of waves supported by a magnetised fluid. Waves 1 and 7 (u_1 and u_7) are *fast magnetosonic* waves, waves 2 and 6 are *Alfvén waves*, waves 3 and 5 are *slow magnetosonic* waves, and wave 4 is an entropy wave which, as the only wave not affected by the presence of a magnetic field, is the only type of wave identical to its hydrodynamic counterpart. It is left as an exercise to show the following inequalities hold:

$$\boxed{\begin{array}{l} a_{\rm s} \le c_{\rm s} \le a_{\rm f}; \\ a_{\rm s} \le a \le a_{\rm f}; \\ a_{\rm s} \le a_x \le a_{\rm f}, \end{array}} \tag{42}$$

and thus, the wave speeds as listed in (41) are in ascending order from the most negative to the most positive.

Alfvén waves propagating along magnetic field lines are like vibrations along telephone wires. They are *transverse* (ordinary sound waves are *longitudinal*), and have no analogue in a non-magnetised fluid. As $B \to 0$ in an MHD fluid, Alfvén waves simply disappear. It was the detection of this type of wave in a laboratory in 1958 that convinced the physics world that MHD was a valid theory.

Fast and slow waves are both compressional and transverse, with both thermal and magnetic pressure variations driving the longitudinal component, and magnetic variations alone driving the transverse component. As $B \to 0$, the slow wave disappears, whereas the fast wave morphs to an ordinary sound wave.

In the atmosphere, a thunder clap is the result of a disturbance (charge transfer between clouds and/or the ground) that launches both sound and entropy waves in all directions[5]. No other waves are launched because the air doesn't support any other wave types.

In an MHD medium, that same thunder clap would generate all four types of waves: entropy, slow, Alfvén, and fast waves. An observer would first hear (yes, hear) the fast wave, then might observe the Alfvén wave as a transverse wind shear (and certainly on a magnetometer), then would hear the slow wave. If it hadn't already completely dissipated, an entropy wave (a gust of wind) might then follow. As the scale of the disturbance would

[5] When I clap my hands, I launch both a sound wave and an "entropy wave", the latter being a small a gust of wind.

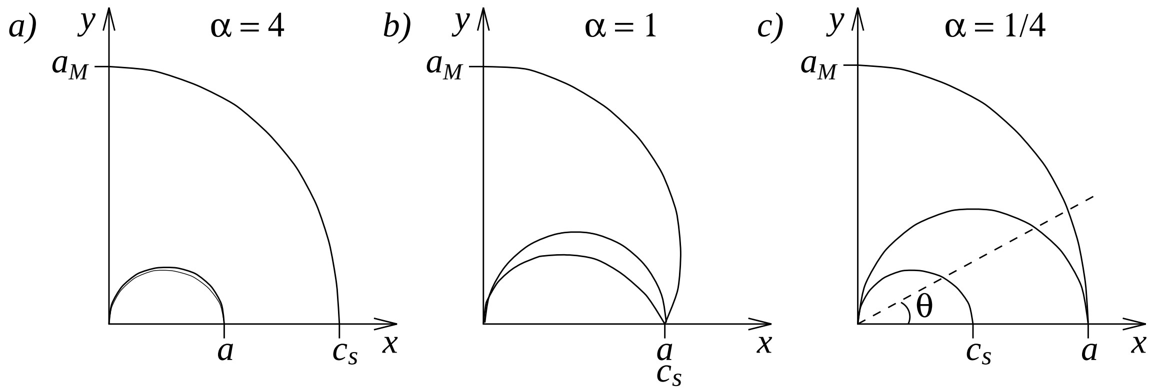

Figure 3: Polar diagrams representing wave speeds of the slow (inner curve), Alfvén (middle curve), and fast (outer curve) waves for $\alpha \equiv c_s^2/a^2 = 4$ (panel *a*), $\alpha = 1$ (panel *b*), and $\alpha = \frac{1}{4}$ (panel *c*). In each case, the wave speed is proportional to the distance between the origin and where a ray inclined at angle θ (angle between $\vec{B}$ and the x-axis) intersects a given curve, as exemplified by the dashed line in panel *c*.

dictate the dominant wavelength of the waves, the fast wave—moving faster—would sound "higher" than the slow wave ($f = v/\lambda$).

Note that the fast and slow speeds are directional. Unlike an ordinary sound wave which is isotropic, both a_s and a_f depend upon $a_x = a\cos\theta$, where θ is the angle between the x-axis (direction of wave propagation) and the orientation of the magnetic field vector (Fig. 3). Note further on Fig. 3 that the slow curve is completely contained by (or touches) the Alfvén curve which is completely contained by (or touches) the fast curve, as required by equation (42).

8 MHD Shocktubes

A *shocktube* is a long tube inside which an essentially 1-D gas experiment can be performed. Typically, one sets two constant states—left and right—with different densities, pressures, *etc.*, separated by a solid diaphragm as shown in Fig. 4. At some prescribed time, the diaphragm is removed and the two states interact. How they interact can be quite dramatic depending on how different the left and right states are initially.

The mathematical equivalent of a laboratory shocktube is called a *Riemann problem*. While there is no record of Riemann ever doing any specific calculations in fluid dynamics, he did develop the mathematics we use in order to find the analytical solution to a shocktube problem. This is beyond the scope of this primer (though determining the eigenvalues of the MHD Jacobian is part of it), and we'll focus here on the phenomenology.

Figure 5 a) shows the initial left and right states of a typical shocktube. The diaphragm located at $x_1 = 0.5$ separates two MHD monatomic ($\gamma = 5/3$) states, both initially at rest ($v_1 = v_2 = 0$) and with $B_1 = 1$. The left state has $\rho = 1$, $p = 1$, and $B_2 = 1$, while the right state has $\rho = 0.2$, $p = 0.1$, and $B_2 = 0$, all in arbitrarily scaled units. At $t = 0$ the

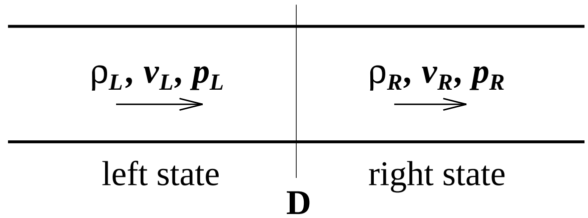

Figure 4: A schematic diagram of a shocktube. At $t = 0$, the diaphragm, $\mathbf{D}$, is removed, and the two constant states interact with an arbitrary jump in (possibly) all flow variables at $\mathbf{D}$.

diaphragm is removed and the two states interact with the semi-analytical Riemann solution of this interaction shown in Fig. 5 b) at $t = 0.15$ in these same scaled units.

The moment the two states are in contact, the jump becomes unstable with the left state (higher pressure) pushing into the right. This interaction sends out all types of waves in both directions with their cumulative effect in both time and space resulting in five distinct features in Fig. 5 b. On the left, fast and slow *rarefaction fans* develop as material from the left "spills" into the right. This phenomenon is similar to what happens in a sandbox divided in two equal halves with sand in one side but not the other. At the instant the divide is

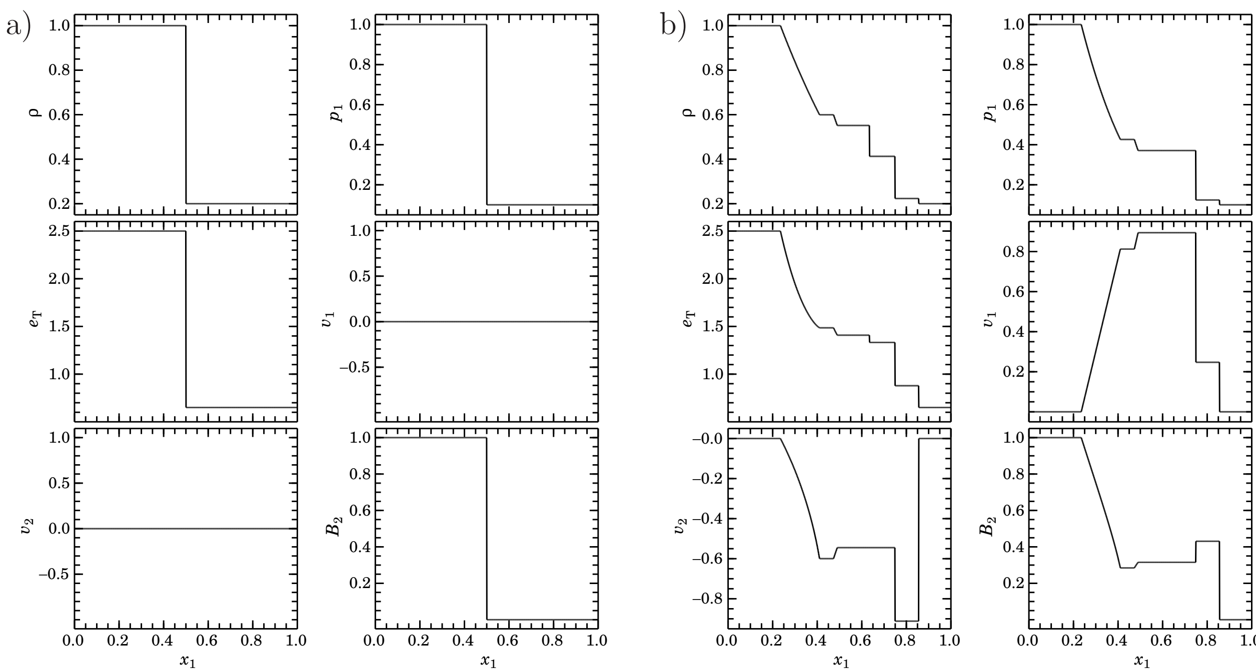

Figure 5: a) Initial state of a typical Riemann problem (adapted from Fig. 4a of Ryu & Jones, 1995, ApJ). b) Final state after $t = 0.15$ showing from left to right: a fast rarefaction, slow rarefaction, contact discontinuity (entropy wave), slow shock, and a fast shock.

removed, the jump in sand depth is vertical but this jump quickly leans over as sand pours from the deep side to the shallow, superficially resembling the rarefaction fans in Fig. 5 b.

At $x_1 \sim 0.64$, an entropy wave (*a.k.a. contact*; a discontinuity in density but not the pressure) separates the original material on the left from that on the right, indicating how far the left half has pushed into the right. An atmospheric example of a contact is a weather front, which is a discontinuity in temperature (and thus density) but not pressure. Cool, dry air moves into a region of hot, moist air causing the latter to rise over the former (cool, dry air is denser, despite our physiological impression). As a result, the moist air cools and loses some of its moisture to precipitation.

Finally, at $x_1 = 0.75$ and 0.86 are, respectively, a slow and fast shock, triggered by the contact moving into the right state faster than either the local slow or fast speeds. Similarly in the atmosphere, anything that tries to move faster than the speed of sound (*e.g.*, a supersonic aircraft) triggers a leading (bow) shock. Shocks are discontinuities in all variables and astrophysically, their passage through the ISM can drive turbulence, excite line emission, even trigger star formation.

We shall revisit this particular shocktube in the *ZEUS-3D* primer.

A Problem Set

1. a) Starting with the continuity equation (9) and the adiabatic equation of state ($p \propto \rho^\gamma$), derive the pressure equation (26).

b) Starting with the continuity equation (9) and the MHD momentum equation (23), derive the MHD Euler equation (25).

2. a) From the pressure equation (26) and the ideal gas law (15), show that the evolution equation for the internal energy is:

$$\frac{\partial e}{\partial t} + \nabla \cdot (e\vec{v}) \;=\; -p\nabla \cdot \vec{v}. \tag{43}$$

b) From equations (9), (20), (25), and (43), derive the evolution equation for the total magnetic energy, (24).

3. By using the closed forms for the MHS wave speeds (equations 40, 39, and 38), prove inequalities (42).

4. The form of the momentum equation actually solved by *ZEUS-3D* for ideal MHD with no self-gravity is:

$$\frac{\partial \vec{s}}{\partial t} + \nabla \cdot \Big(\vec{s}\,\vec{v} + (p + p_B)\,\mathsf{I} - \vec{B}\vec{B}\Big) \;=\; 0, \tag{44}$$

where $\vec{s}\,\vec{v}$ and $\vec{B}\vec{B}$ are dyadic products of the vectors, I is the identity tensor (matrix), and

$p_B = B^2/2\mu_0$ is the magnetic pressure. Using the vector identities in Appendix B, show how this may be derived from equation (23).

B Vector Identities

Let $\vec{A}$, $\vec{B}$, $\vec{C}$, and $\vec{D}$ be four arbitrary vectors. Then,

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{B}\cdot(\vec{C}\times\vec{A}) = \vec{C}\cdot(\vec{A}\times\vec{B}); \tag{45}$$

$$\vec{A}\times(\vec{B}\times\vec{C}) = (\vec{A}\cdot\vec{C})\vec{B} - (\vec{A}\cdot\vec{B})\vec{C}; \tag{46}$$

$$(\vec{A}\times\vec{B})\cdot(\vec{C}\times\vec{D}) = (\vec{A}\cdot\vec{C})(\vec{B}\cdot\vec{D}) - (\vec{A}\cdot\vec{D})(\vec{B}\cdot\vec{C}). \tag{47}$$

Let f and g be two arbitrary scalar functions of the coordinates, and let $\vec{A}$ and $\vec{B}$ be two arbitrary vector functions of the coordinates. Then,

$$\nabla(fg) = f\nabla g + g\nabla f; \tag{48}$$

$$\nabla(f/g) = \frac{g\nabla f - f\nabla g}{g^2}; \tag{49}$$

$$\nabla(\vec{A}\cdot\vec{B}) = (\vec{B}\cdot\nabla)\vec{A} + (\vec{A}\cdot\nabla)\vec{B} + \vec{B}\times(\nabla\times\vec{A}) + \vec{A}\times(\nabla\times\vec{B}); \tag{50}$$

$$\nabla\cdot(f\vec{A}) = f\nabla\cdot\vec{A} + \vec{A}\cdot\nabla f; \tag{51}$$

$$\nabla\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\nabla\times\vec{A}) - \vec{A}\cdot(\nabla\times\vec{B}); \tag{52}$$

$$\nabla\times(f\vec{A}) = f\nabla\times\vec{A} + \nabla f\times\vec{A}; \tag{53}$$

$$\nabla\times(\vec{A}\times\vec{B})^{6} = (\vec{B}\cdot\nabla)\vec{A} - (\vec{A}\cdot\nabla)\vec{B} - \vec{B}(\nabla\cdot\vec{A}) + \vec{A}(\nabla\cdot\vec{B}); \tag{54}$$

$$\nabla\times(\nabla f) = 0; \tag{55}$$

$$\nabla\cdot(\nabla\times\vec{A}) = 0. \tag{56}$$

For $\nabla\times(\nabla\times\vec{A})$, see identity (69).

Identities (57) and (58) below are particularly useful for working with the MHD equations and can be derived from the more fundamental identities above.

$$\begin{aligned}(\vec{A}\cdot\nabla)\vec{B} = \frac{1}{2}\Big[&\nabla(\vec{A}\cdot\vec{B}) + \vec{A}(\nabla\cdot\vec{B}) - \vec{B}(\nabla\cdot\vec{A}) - \nabla\times(\vec{A}\times\vec{B}) \\ &- \vec{A}\times(\nabla\times\vec{B}) - \vec{B}\times(\nabla\times\vec{A})\Big];\end{aligned} \tag{57}$$

$$(\vec{A}\cdot\nabla)\vec{A} = \tfrac{1}{2}\nabla A^2 - \vec{A}\times(\nabla\times\vec{A}). \tag{58}$$

[6] $\nabla\times(\vec{A}\times\vec{B})$ may also be expressed in terms of perfect divergences; see identity (65).

Constructs such as $\vec{A}\vec{B}$ (the *dyadic product* of the vectors $\vec{A}$ and $\vec{B}$) as well as $\nabla\vec{A}$ (the gradient of the vector $\vec{A}$) are examples of *dyadics* (and thus tensors) and appear frequently in the MHD equations. In Cartesian coordinates, these look like:

$$\vec{A}\vec{B} = |A\rangle\langle B| = \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} [B_x \; B_y \; B_z] = \begin{bmatrix} A_xB_x & A_xB_y & A_xB_z \\ A_yB_x & A_yB_y & A_yB_z \\ A_zB_x & A_zB_y & A_zB_z \end{bmatrix}; \tag{59}$$

$$\nabla\vec{A} = \begin{bmatrix} \partial_xA_x & \partial_xA_y & \partial_xA_z \\ \partial_yA_x & \partial_yA_y & \partial_yA_z \\ \partial_zA_x & \partial_zA_y & \partial_zA_z \end{bmatrix}. \tag{60}$$

The *colon product* (double contraction) of two dyadics $\mathsf{M} = \vec{A}\vec{B}$ and $\mathsf{N} = \vec{C}\vec{D}$ is defined as:

$$\mathsf{M} : \mathsf{N} \equiv \sum_{ij} M_{ij}N_{ij} = \sum_{ij} A_iB_jC_iD_j = \sum_i A_iC_i \sum_j B_jD_j = (\vec{A}\cdot\vec{C})(\vec{B}\cdot\vec{D}). \tag{61}$$

Following are several useful identities involving the tensors $\nabla\vec{A}$, $\vec{A}\vec{B}$, and T, the latter not necessarily a dyadic. The superscript $^{\mathrm{T}}$ denotes the *transpose* of a tensor in which, in its matrix representation, the rows of the tensor form the columns of the transpose. Thus, $(\vec{A}\vec{B})^{\mathrm{T}} = \vec{B}\vec{A}$. The first equality in identity (64) is analogous to (51) while the second two follow from (57). Identities (65) and (66) then follow trivially from (64).

$$\vec{A}\cdot(\nabla\vec{A}) = (\vec{A}\cdot\nabla)\vec{A} = \tfrac{1}{2}\nabla A^2 - \vec{A}\times(\nabla\times\vec{A}); \tag{62}$$

$$\vec{A}\cdot(\nabla\vec{A})^{\mathrm{T}} = \tfrac{1}{2}\nabla A^2; \tag{63}$$

$$\left.\begin{aligned} \nabla\cdot(\vec{A}\,\vec{B}) &= (\vec{A}\cdot\nabla)\vec{B} + \vec{B}(\nabla\cdot\vec{A}) \\ &= \frac{1}{2}\Big[\nabla(\vec{A}\cdot\vec{B}) + \vec{A}(\nabla\cdot\vec{B}) + \vec{B}(\nabla\cdot\vec{A}) - \nabla\times(\vec{A}\times\vec{B}) \\ &\qquad - \vec{A}\times(\nabla\times\vec{B}) - \vec{B}\times(\nabla\times\vec{A})\Big] \\ &= (\vec{B}\cdot\nabla)\vec{A} + \vec{A}(\nabla\cdot\vec{B}) - \nabla\times(\vec{A}\times\vec{B}); \end{aligned}\right\} \tag{64}$$

$$\nabla\cdot(\vec{A}\,\vec{B}) = \nabla\cdot(\vec{B}\,\vec{A}) - \nabla\times(\vec{A}\times\vec{B}); \tag{65}$$

$$\left.\begin{aligned} \nabla\cdot(\vec{A}\,\vec{A}) &= (\vec{A}\cdot\nabla)\vec{A} + \vec{A}(\nabla\cdot\vec{A}) \\ &= \tfrac{1}{2}\nabla A^2 + \vec{A}(\nabla\cdot\vec{A}) - \vec{A}\times(\nabla\times\vec{A}); \end{aligned}\right\} \tag{66}$$

$$\nabla\cdot(\mathsf{T}\cdot\vec{A}) = \mathsf{T} : \nabla\vec{A} + (\nabla\cdot\mathsf{T})\cdot\vec{A}; \tag{67}$$

$$\nabla\vec{A} : \nabla\vec{A} = (\nabla\times\vec{A})\cdot(\nabla\times\vec{A}) + \nabla\vec{A} : (\nabla\vec{A})^{\mathrm{T}}; \tag{68}$$

$$\nabla\cdot(\nabla\vec{A}) = \nabla^2\vec{A} = \nabla(\nabla\cdot\vec{A}) - \nabla\times(\nabla\times\vec{A}); \tag{69}$$

$$\nabla\cdot(\nabla\vec{A})^{\mathrm{T}} = \nabla(\nabla\cdot\vec{A}). \tag{70}$$